Nash Equilibria for

$$\text{Two-Player Matrix Games } \underbrace{\text{Repeated Until Collison}}_{\text{aka RUC}}$$

Two players ⇝

Batter & Bowler

Game proceeds in multiple rounds.

Two players ⇝

Batter & Bowler

In each round, the players simultaneously pick numbers from

$\{0, 1, 2, 3, 4, 5, 6\}$

Two players $\rightsquigarrow$

Batter & Bowler

In each round, the players simultaneously pick numbers from

$\{0, 1, 2, 3, 4, 5, 6\}$

Two players $\rightsquigarrow$

Batter & Bowler

2

In each round, the players simultaneously pick numbers from

$\{0, 1, 2, 3, 4, 5, 6\}$

Two players $\rightsquigarrow$

Batter & Bowler

2

In each round, the players simultaneously pick numbers from

$\{0, 1, 2, 3, 4, 5, 6\}$

Two players ⇝

Batter & Bowler

2+5

In each round, the players simultaneously pick numbers from

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Two players $\rightsquigarrow$

Batter & Bowler

2+5

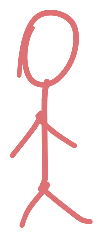
In each round, the players simultaneously pick numbers from

$\{0,1,2,3,4,5,6\}$

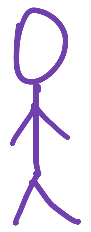
Two players $\rightsquigarrow$

Batter & Bowler

$2+5+0$

 3

Game Over!

 3

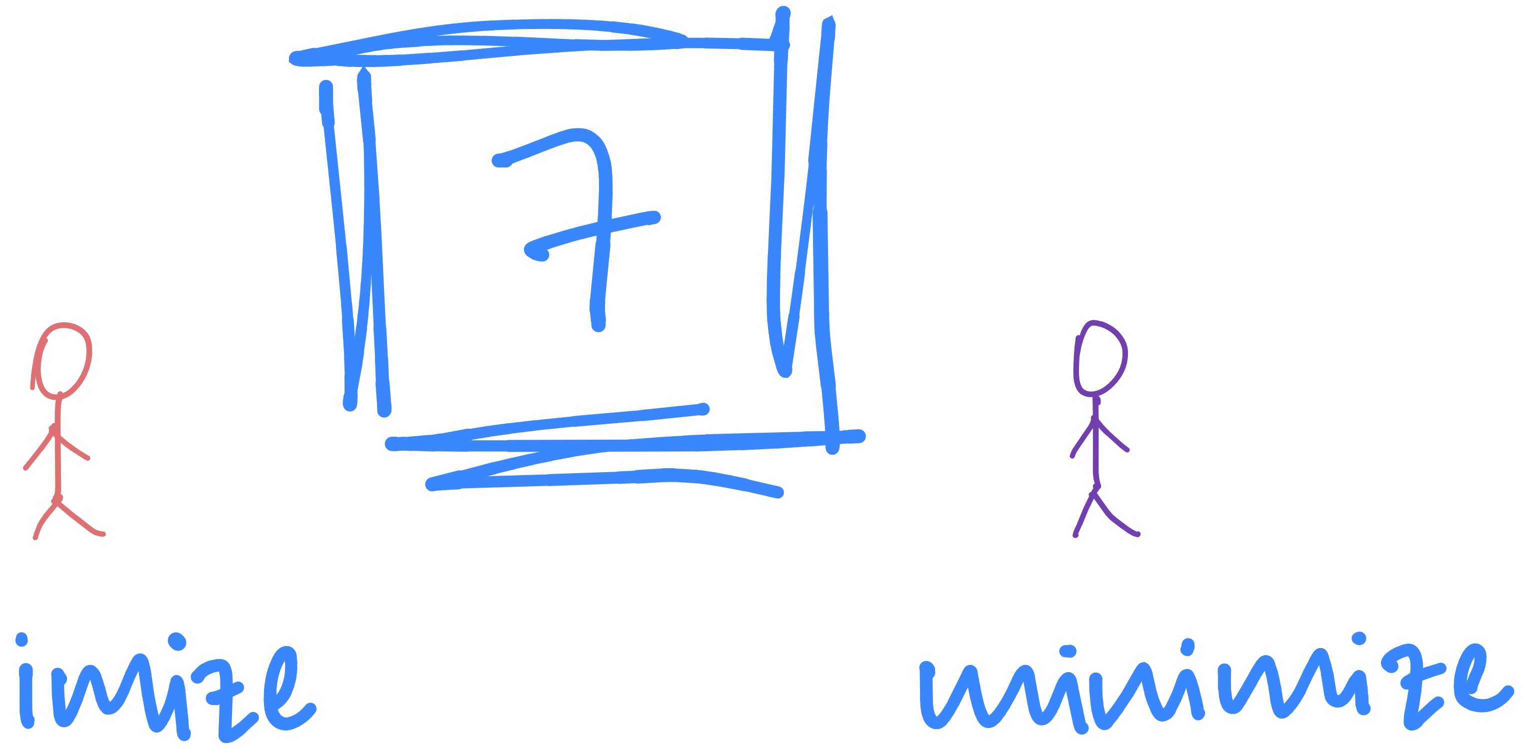
In each round, the players simultaneously pick numbers from

$\{0,1,2,3,4,5,6\}$

Two players $\rightsquigarrow$

Batter & Bowler

$2+5+0$



In each round, the players simultaneously pick numbers from

$\{0, 1, 2, 3, 4, 5, 6\}$

Max Player

Min Player

$$\underbrace{A \qquad\qquad\qquad B}_{n \times n \text{ matrices}}$$

multiple rounds

Each round : Both players pick a number from $\{1, 2, \ldots, n\}$

Max Player (p)

Min Player (q)

max player earns a SCORE of A_{ij}

min player incurs a COST of B_{ij}

Termination Condition

$p = q$

(collison)

Hand Cricket

$$A_{ij} = B_{ij} = \begin{cases} i & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Intuitively - how to win a RUC game?

Max player wants the game to last long.

Min player wants the game to end soon.

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MAX

Intuitively - how to win a RUC game?

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MAX

1 Pick all actions uniformly at random

Intuitively - how to win a RUC game?

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MAX

1 Pick all actions uniformly at random

2. Play high-scoring moves more often

Intuitively - how to win a RUC game?

Max player wants the game to last long.

Min player wants the game to end soon.

MAX

1 Pick all actions uniformly at random

good for duration

2. Play high-scoring moves more often

good for score

NASH EQUILIBRIA

Two player single-round games

(x^*, y^*) is NE if BOTH of the following hold :

$\forall x \in \mathcal{X}, \quad f_1(x, y^*) \leq f_1(x^*, y^*)$ (no benefit $x^* \to x$)

$\forall y \in \mathcal{Y}, \quad f_2(x^*, y) \geq f_2(x^*, y^*)$ (no benefit $y^* \to y$)

NASH EQUILIBRIA

Two player <u>single</u>-round games

Strategy space

(probability vectors)

$$\Delta_n := \{ x \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n = 1 \}$$

Let $A, B \in \mathbb{R}^{m \times n}$ and $\mathcal{X} = \mathcal{Y} = \Delta_n$.

$f_1(x,y) = x^T A y$ and $f_2(x,y) = x^T B y$

NASH EQUILIBRIA

Two player single-round games

$$\Delta_n := \{ x \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n = 1 \}$$

Let $A, B \in \mathbb{R}^{m \times n}$ and $\mathcal{X} = \mathcal{Y} = \Delta_n$.

$f_1(x,y) = x^T A y$ and $f_2(x,y) = x^T B y$

A Nash Equilibrium always exists

NE for RUC games.

$\boxed{\text{NE always* exists for RUC games}}$

* under reasonable assumptions about A and B.

NE for RUC games.

NE always* exists for RUC games

ξ^e is "almost" unique.

* under reasonable assumptions about A and B.

NE for Stationary RUC games

Let $x \in \Delta_n$. The stationary strategy x:

In each round, pick action i with probability x_i.

(Actions are picked independently in each round)

SRUC games $\rightsquigarrow$ players can only play stationary strategies.

NE for Stationary RUC games.

Max Player

x

Min Player

y

NE for Stationary RUC games.

Max Player x

Min Player y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

NE for Stationary RUC games.

Max Player x

Min Player y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

$$f_1(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n}$$

NE for Stationary RUC games.

Max Player x

Min Player y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

$$f_1(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j$$

NE for Stationary RUC games.

Max Player x

Min Player y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

$$f_1(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \Big(A(i,j)$$

NE for Stationary RUC games.

Max Player x

Min Player y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

$$f_1(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \Big(A(i,j) + f_1(x,y)\, \mathbb{1}[i \neq j] \Big)$$

NE for Stationary RUC games.

Max Player: x

Min Player: y

$f_1(x,y) \rightsquigarrow$ player 1's total expected score

$$f_1(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \Big(A(i,j) + f_1(x,y)\, \mathbb{1}[i \neq j] \Big)$$

$$= x^T A y + f_1(x,y)\left(1 - x^T y\right)$$

NE for Stationary RUC games.

Max Player x

Min Player y

$$f_1(x,y) = \frac{x^T A y}{x^T y}$$

$$f_2(x,y) = \frac{x^T B y}{x^T y}$$

NE for Stationary RUC games.

Max Player

x

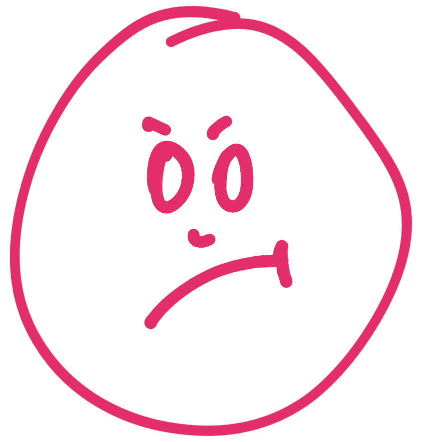
Min Player

y

$$f_1(x,y) = \frac{x^T A y}{x^T y}$$

$$f_2(x,y) = \frac{x^T B y}{x^T y}$$

not bilinear



NE for Stationary RUC games.

Max Player x

Min Player y

Example

$$A = B = \begin{pmatrix} 0 & s_1 \\ s_2 & 0 \end{pmatrix}$$

NE for (Stationary) RUC games.

Max Player ... x

Min Player ... y

Example

$$A = B = \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}$$

Turns out:

$$x^* = \left(\frac{\sqrt{S_2}}{\alpha}, \frac{\sqrt{S_1}}{\alpha} \right)$$

$$\alpha = \sqrt{S_1} + \sqrt{S_2}$$

$$y^* = \left(\frac{\sqrt{S_2}}{\alpha}, \frac{\sqrt{S_1}}{\alpha} \right)$$

NE for Stationary RUC games.

Max Player $\circ\circ\circ$ x

Min Player $\circ\circ\circ$ y

Example

$$A = B = \begin{pmatrix} 0 & 1 \\ 10^4 & 0 \end{pmatrix}$$

Turns out .

$$x^* = \left(\frac{\sqrt{S_2}}{\alpha}, \frac{\sqrt{S_1}}{\alpha} \right)$$

$$\alpha = \sqrt{S_1} + \sqrt{S_2}$$

$$y^* = \left(\frac{\sqrt{S_2}}{\alpha}, \frac{\sqrt{S_1}}{\alpha} \right)$$

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

S'pose A has an eigenpair

$$(\lambda_A, y^*)$$

s.t. $y^* \in \Delta_n$.

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

S'pose A has an eigenpair

$$(\lambda_A, y^*)$$

$$(A \cdot y^* = \lambda_A y^*)$$

s.t. $y^* \in \Delta_n$.

NE for Stationary RUC games

Max Player

x

Min Player

y

S'pose B^T has an eigenpair

$$(\lambda_B, x^*)$$

$$(B^T \cdot x^* = \lambda_B x^*)$$

s.t. $x^* \in \Delta_n$.

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

$$f_1(x, y^*) = \frac{x^T A y^*}{x^T y^*}$$

$$f_2(x^*, y) = \frac{x^{*T} B y}{x^{*T} y}$$

NE for Stationary RUC games

Max Player $\ldots x$

Min Player $\ldots y$

$$f_1(x, y^*) = \frac{x^T A y^*}{x^T y^*} = \lambda_A$$

$$f_2(x^*, y) = \frac{x^{*T} B y}{x^{*T} y} = \lambda_B$$

NE for Stationary RUC games.

Max Player x

Min Player y

independent of x

$$f_1(x, y^*) = \frac{x^T A y^*}{x^T y^*} = \lambda_A$$

$$f_2(x^*, y) = \frac{{x^*}^T B y}{{x^*}^T y} = \lambda_B$$

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

When do A, B have eigenvectors that are prob. distributions?

(Detour)

Perron-Frobenius Theorem

(Detour)

Perron-Frobenius Theorem

Matrix M $(n \times n)$ $\rightarrow$ Graph G_M

(Detour)

Perron-Frobenius Theorem

Matrix M $\longrightarrow$ Graph G_M

$(n \times n)$

$$V(G_M) = \{1, 2, \dots, n\}$$

(Detour)

Perron-Frobenius Theorem

Matrix $\underset{(n \times n)}{M} \longrightarrow$ Graph G_M

$$V(G_M) = \{1, 2, \ldots, n\}$$

$$E(G_M) = \{(i,j) \mid M_{ij} \neq 0\}$$

$$\ell((i,j)) = M_{ij}$$

(Detour)

Perron-Frobenius Theorem

Matrix M $\rightarrow$ Graph G_M

$(n \times n)$

$$\begin{pmatrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$V(G_M) = \{1, 2, \dots, n\}$$

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Matrix M $(n \times n)$ $\rightarrow$ Graph G_M

$$\begin{array}{c c} & \begin{array}{cccc} \star & \star & \star & \star \end{array} \\ \begin{array}{c} \star \\ \star \\ \star \\ \star \end{array} & \begin{pmatrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

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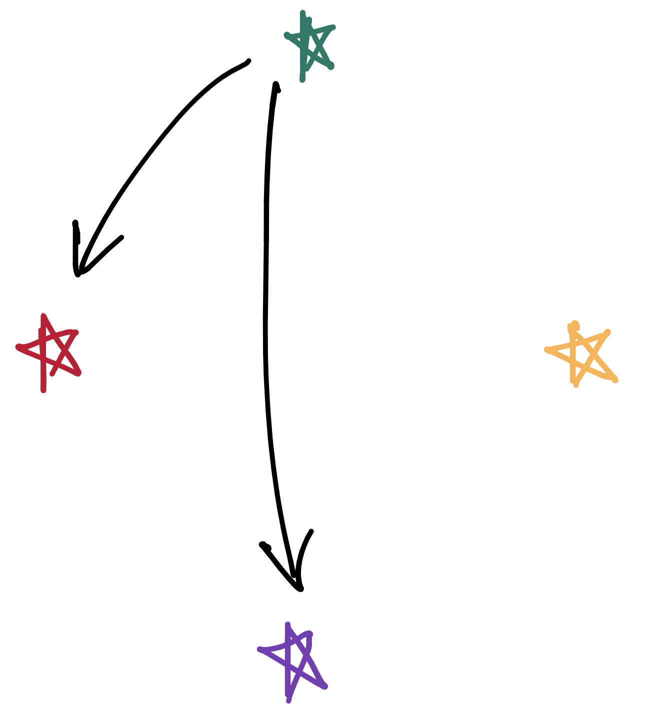

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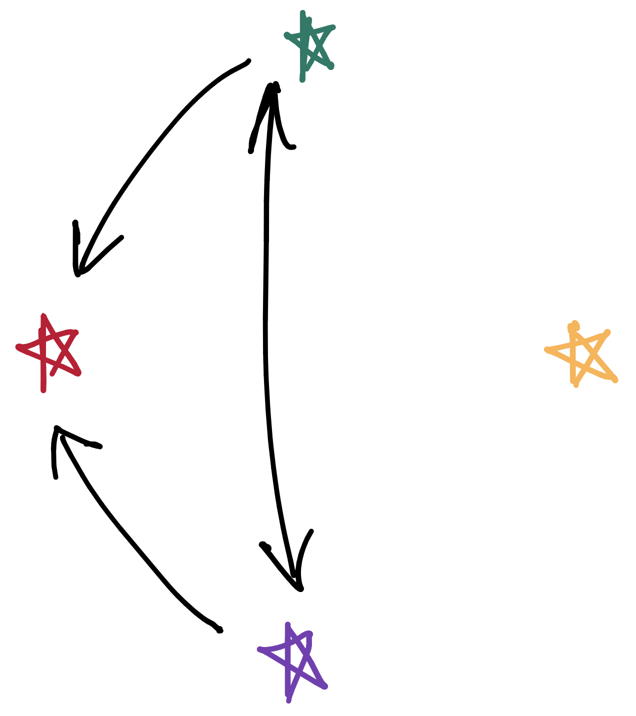


(Detour)

Perron-Frobenius Theorem

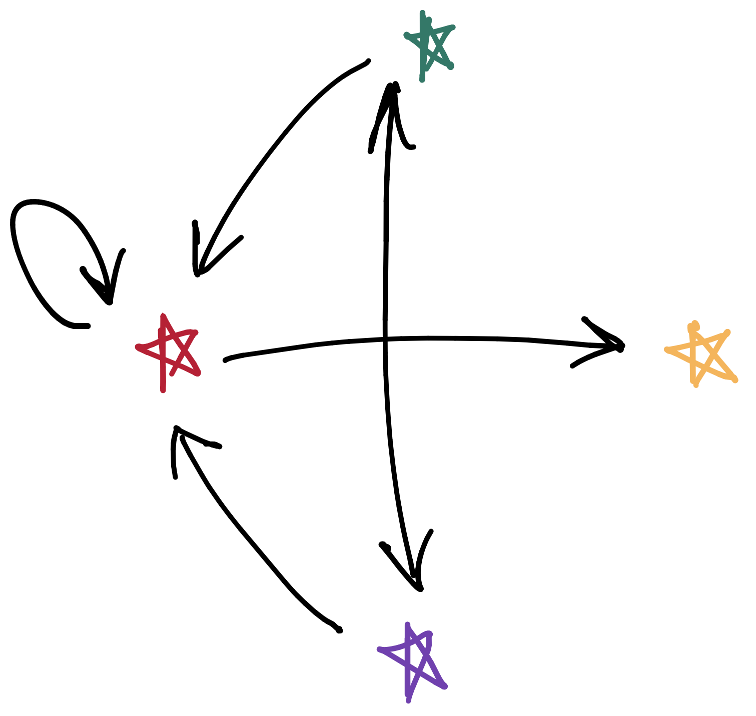
Matrix M $\longrightarrow$ Graph G_M
$(n \times n)$

$$
\begin{array}{c|cccc}
 & \star & \star & \star & \star \\
\hline
\star & & & & \\
\star & & & & \\
\star & & & & \\
\star & & & &
\end{array}
\quad
\begin{pmatrix}
0 & 3 & 8 & 0 \\
5 & 0 & 4 & 0 \\
0 & 0 & 2 & 1 \\
0 & 1 & 2 & 0
\end{pmatrix}
$$

$$V(G_M) = \{1, 2, \ldots, n\}$$

$$E(G_M) = \{(i,j) \mid M_{ij} \neq 0\}$$

$$\ell((i,j)) = M_{ij}$$



(Detour)

Perron-Frobenius Theorem

Matrix M $(n \times n)$ $\longrightarrow$ Graph G_M

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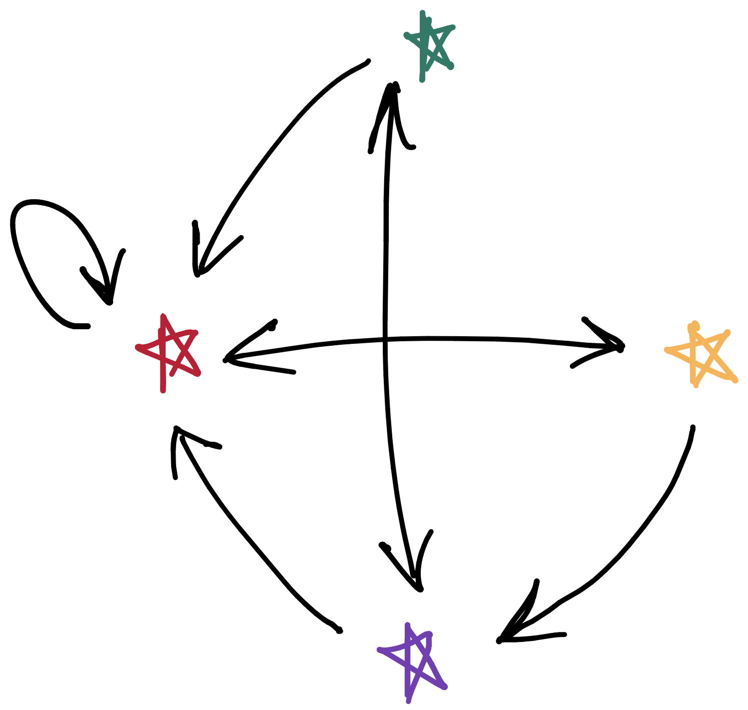
$$E(G_M) = \{(i,j) \mid M_{ij} \neq 0\}$$

$$\ell((i,j)) = M_{ij}$$

$$\begin{array}{c|cccc} & \star & \star & \star & \star \\ \hline \star & & & & \\ \star & & \left(\begin{matrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{matrix}\right) & & \\ \star & & & & \\ \star & & & & \end{array}$$



(Detour)

Perron-Frobenius Theorem

Matrix M $\rightarrow$ Graph G_M

$(n \times n)$

irreducible $\qquad\qquad$ G_M is strongly connected

$$\begin{array}{c c} & \begin{array}{cccc} \star & \star & \star & \star \end{array} \\ \begin{array}{c} \star \\ \star \\ \star \\ \star \end{array} & \begin{pmatrix} 0 & 3 & 8 & 0 \\ 5 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

$$V(G_M) = \{1, 2, \ldots, n\}$$

$$E(G_M) = \{(i,j) \mid M_{ij} \neq 0\}$$

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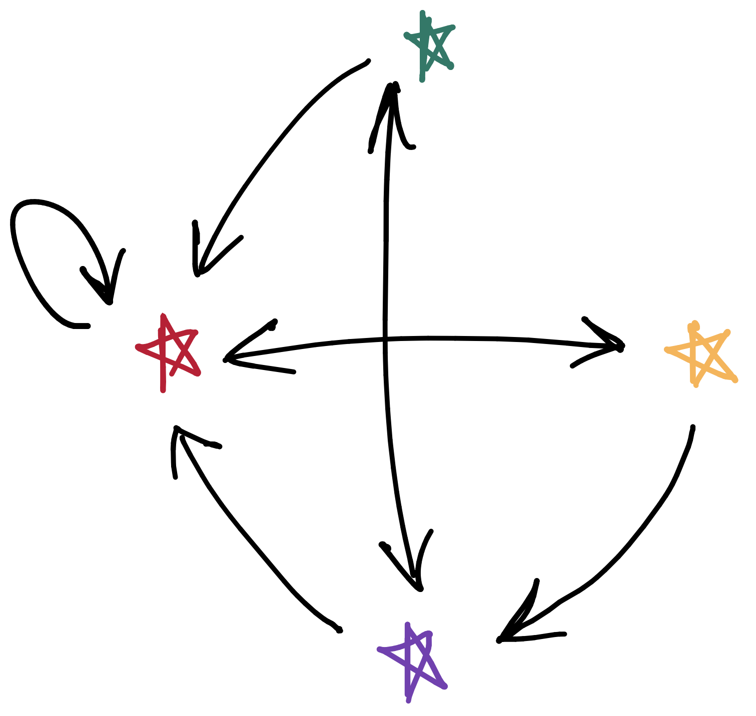


(Detour)

Perron-Frobenius Theorem

Let M be an irreducable matrix w/ entries in $\mathbb{N}_0$.

(Detour)

Perron-Frobenius Theorem

Let M be an irreducable matrix w/ entries in $\mathbb{N}_0$.

1. There is a unique eigenvalue $\lambda^* \in \mathbb{R}_{>0}$

   whose absolute value is bigger than all other eigenvalues.

(Detour)

Perron-Frobenius Theorem

Let M be an irreducable matrix w/ entries in $\mathbb{N}_0$.

2. There exist unique vectors $u, v \in \mathbb{R}^n_{>0}$ s.t.

$$\sum u_i = 1 = \sum v_i \quad \text{and}$$

$$A^T u = \lambda^* \cdot u \quad \text{and} \quad Av = \lambda^* \cdot v$$

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

When do A, B have eigenvectors that are prob. distributions?

When A & B are IRREDUCIBLE.

NE for Stationary RUC games.

Max Player ... x

Min Player ... y

When do A, B have eigenvectors that are prob. distributions?

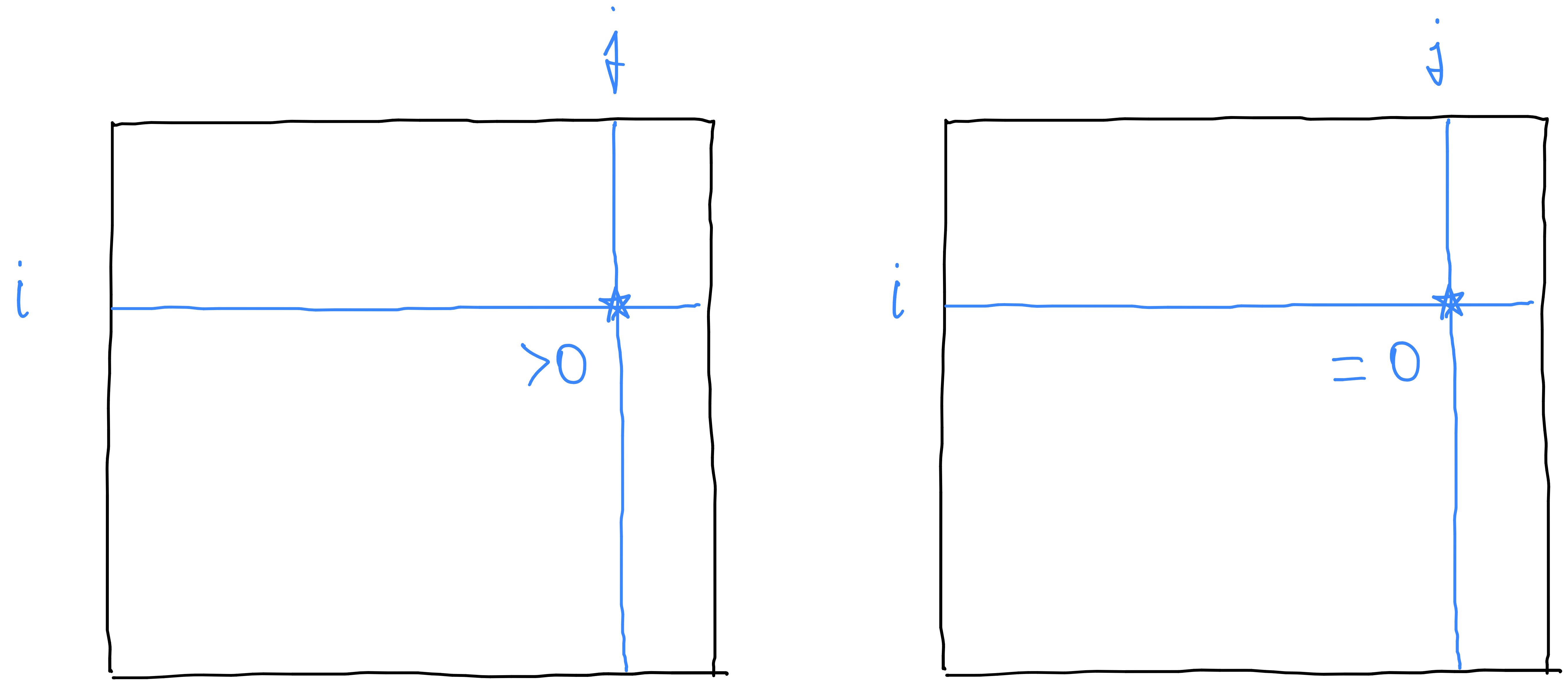
What about when they are not?

NE for Stationary RUC games.

Max Player

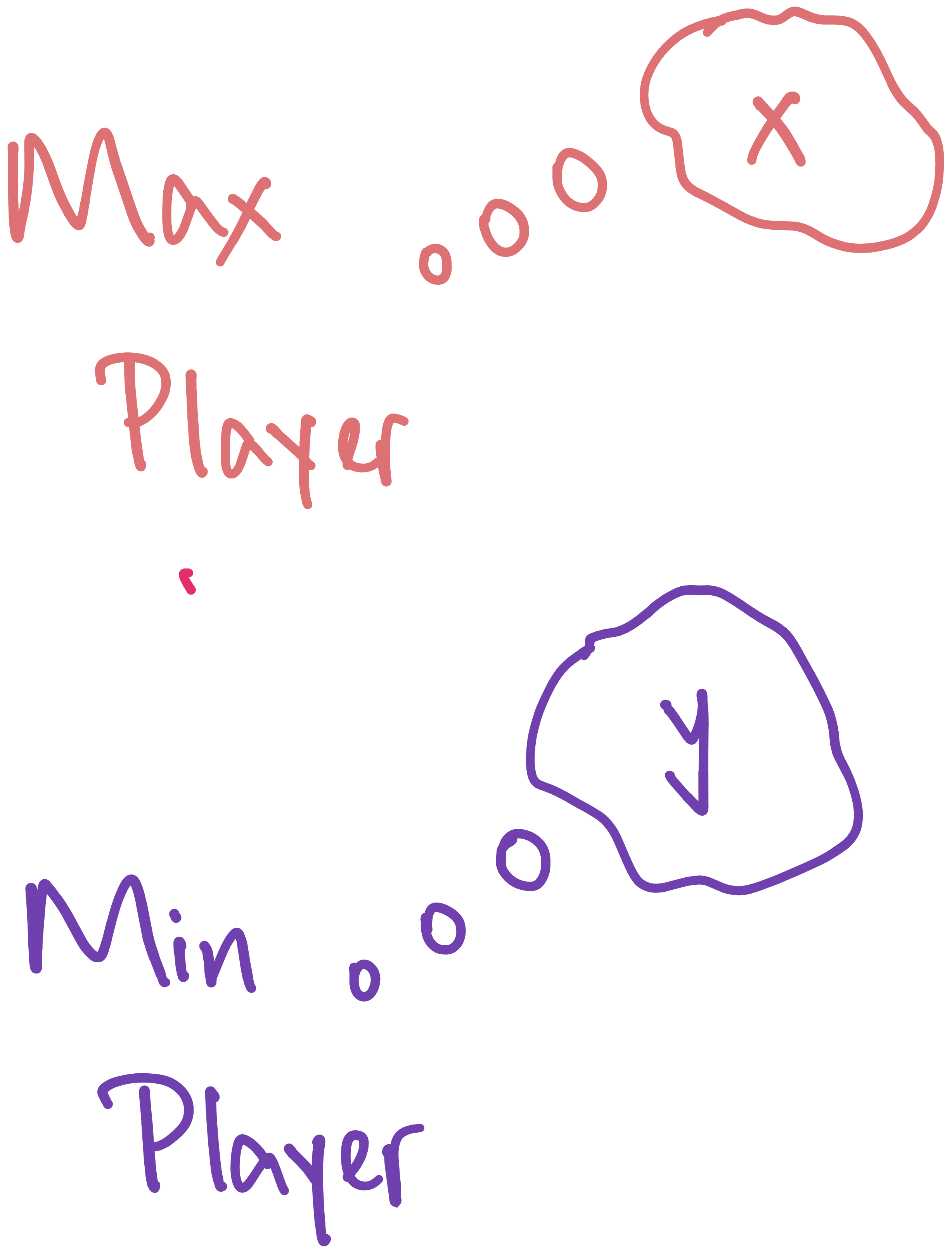
Min Player

Let A, B be s.t. $\exists\, i \neq j$ where:



A   B

NE for Stationary RUC games.



If $G_B \nsubseteq G_A \rightsquigarrow$ deterministic NE.