Games & Equilibria

KICK / DIVE	Left	Right
Left		
Right		

Penalty Shot Game

Games & Equilibria

DIVE \ KICK	Left	Right
Left	+1 -1	
Right		

Penalty Shot Game

Games & Equilibria

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
Right		

Penalty Shot Game

Games & Equilibria

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
Right	−1 +1	

Penalty Shot Game

Games & Equilibria

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
Right	-1 +1	+1 -1

Penalty Shot Game

Games & Equilibria

Equilibria :

A pair of strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
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Penalty Shot Game

Games & Equilibria

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Left	+1 -1	-1 +1
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Penalty Shot Game

Eg. Goal keeper plays LEFT & the opponent plays RIGHT.

Games & Equilibria

Equilibria :

A pair of strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
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Penalty Shot Game

Eg. Goal keeper plays LEFT & the opponent plays RIGHT.

Games & Equilibria

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A pair of strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
Right	−1 +1	+1 −1

Penalty Shot Game

Eg. Goal keeper plays LEFT & the opponent plays RIGHT.

the goalkeeper has an incentive to deviate.

Games & Equilibria

Equilibria:

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
Right	−1 +1	+1 −1

Penalty Shot Game

Games & Equilibria

Equilibria :

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
Right	-1 +1	+1 -1

Penalty Shot Game

Play LEFT w/ prob. 1/2
& RIGHT w/ prob 1/2.

opponent

Games & Equilibria

(RIGHT $\rightsquigarrow$ p, LEFT $\rightsquigarrow$ 1-p)

goalkeeper

Play LEFT w/ prob. 1/2

& RIGHT w/ prob 1/2.

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Left	+1 -1	-1 +1
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Penalty Shot Game

Games & Equilibria

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A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
Right	−1 +1	+1 −1

Penalty Shot Game

(RIGHT w/ p, LEFT w/ 1−p)

goalkeeper

Play LEFT w/ prob. 1/2
& RIGHT w/ prob 1/2.

opponent

Expected payoff :

Games & Equilibria

Equilibria:

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 +1
Right	−1 +1	+1 −1

Penalty Shot Game

(RIGHT w/ p, LEFT w/ 1-p)

goalkeeper

play LEFT w/ prob. 1/2

& RIGHT w/ prob 1/2.

opponent

Expected payoff:

RL RR

LL LR

Games & Equilibria

Equilibria:

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
Right	-1 +1	+1 -1

Penalty Shot Game

(RIGHT w/ p, LEFT w/ $1-p$)

goalkeeper

Play LEFT w/ prob. 1/2 & RIGHT w/ prob 1/2.

opponent

Expected payoff:

RL -1

RR +1

LL +1

LR -1

Games & Equilibria

(RIGHT w.p. p, LEFT w.p. $1-p$)

goalkeeper

Equilibria:

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

DIVE \ KICK	Left	Right
Left	+1 -1	-1 +1
Right	-1 +1	+1 -1

Penalty Shot Game

play LEFT w/ prob. 1/2 & RIGHT w/ prob 1/2.

opponent

Expected payoff:

$$\frac{p}{2}\overset{RL}{(-1)} + \frac{p}{2}\overset{RR}{(+1)}$$

$$\frac{q}{2}\overset{LL}{(+1)} + \frac{q}{2}\overset{LR}{(-1)}$$

$(q = 1-p)$

Games & Equilibria

Equilibria :

A pair of randomized strategies so that no player has incentive to deviate if the other player stays put.

KICK / DIVE	Left	Right
Left	+1 -1	-1 +1
Right	-1 +1	+1 -1

Penalty Shot Game

(RIGHT ⤳ p, LEFT ⤳ 1-p)

goalkeeper

Play LEFT w/ prob. 1/2
& RIGHT w/ prob 1/2.

opponent

Expected payoff :

$$\left.\begin{array}{l} \frac{p}{2}\,\overset{RL}{(-1)} + \frac{p}{2}\,\overset{RR}{(+1)} \\ \frac{q}{2}\,\overset{LL}{(+1)} + \frac{q}{2}\,\overset{LR}{(-1)} \end{array}\right. = 0$$

players $\rightsquigarrow [n]$

$\forall p \rightsquigarrow S_p$ (strategies available to player p)

$$\mathcal{U}_p : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

$$\uparrow$$

$$\mathcal{U}_p(s_1, s_2, \ldots, s_p, \ldots, s_n)$$

$$\downarrow$$

$2(p+x) - 4px - 1$

Left : p Right : q $p+q=1$ S'pose $x = \frac{1}{2} \rightsquigarrow 0$

Left : x Right : y $x+y=1$ S'pose $x = 1/4$ $2p - p - \frac{1}{2} = p - 1/2$

$p = 3/4$

$\Rightarrow 1/4$

$$E(\text{payoff}) = px(-1) + py(+1) + qx(+1) + qy(-1)$$
$$= -px + p(1-x) + (1-p)x - (1-p)(1-x)$$
$$= (p+x) - 3px - \{1 - p - x + px\}$$
$$= 2(p+x) - 4px - 1$$

Randomized Strategy

$S_p \rightsquigarrow$ strategies available to player p.

$\Delta(S_p) \rightsquigarrow$ set of all distributions over S_p.

$$S_p = \{ a, b, c \}$$

$$\Delta(S_p) = \{ \langle p, q, r \rangle \mid p + q + r = 1 \}$$

Expected Utilities

$$U_p(x_1, \ldots, x_p, \ldots, x_n) = \mathop{\mathbb{E}}_{\substack{s_1 \sim x_1 \\ \vdots \\ s_n \sim x_n}} U_p(s_1, \ldots, s_n)$$

$x_p \in \Delta(S_p)$

Nash Equilibrium

A collection $(x_1, \ldots, x_n)$ is a NASH EQUILIBRIUM

iff $\forall p.\quad U_p(x_1, \ldots, x_n) \geqslant U_p(x_1, \ldots, x_p', \ldots, x_n)$

$\forall x_p' \in \Delta(S_p)$

(assuming q plays x_q $\forall q \neq p$,
player p has no reason to not play x_p.)

DIVE \ KICK	Left	Right
Left	+1 −1	−1 −1
Right	−1 +1	+1 −1

two-player zero-sum game

DIVE \ KICK	Left	Right
Left	+1 −1	−1 −1
Right	−1 +1	+1 −1

(von Neumann, '28) A Nash Equilibrium always exists for two-player zero-sum games.

DIVE \ KICK	Left	Right
Left	+1 -1	-1 -1
Right	-1 +1	+1 -1

min-max equilibrium

(von Neumann, '28) A Nash Equilibrium always exists for two-player zero-sum games

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(Nash, '50) An equilibrium exists for every game

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Left	+1 −1	−1 −1
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min-max equilibrium

(von Neumann, '28) A Nash Equilibrium always exists for two-player zero-sum games

(Nash, '50) An equilibrium exists for every* game (* finite # of players)

(Brouwer, 1910) Let $f : D \to D$ be a continuous function from a convex and compact subset D

(closed & bounded)

of the Euclidean space to itself.

Then $\exists$ an $x \in D$ such that $f(x) = x$

Brouwer $\Rightarrow$ Nash.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 −1
Right	−1 +1	+1 −1

Penalty Shot Game

$\longmapsto$

$f : [0,1]^2 \to [0,1]^2$

continuous, so that

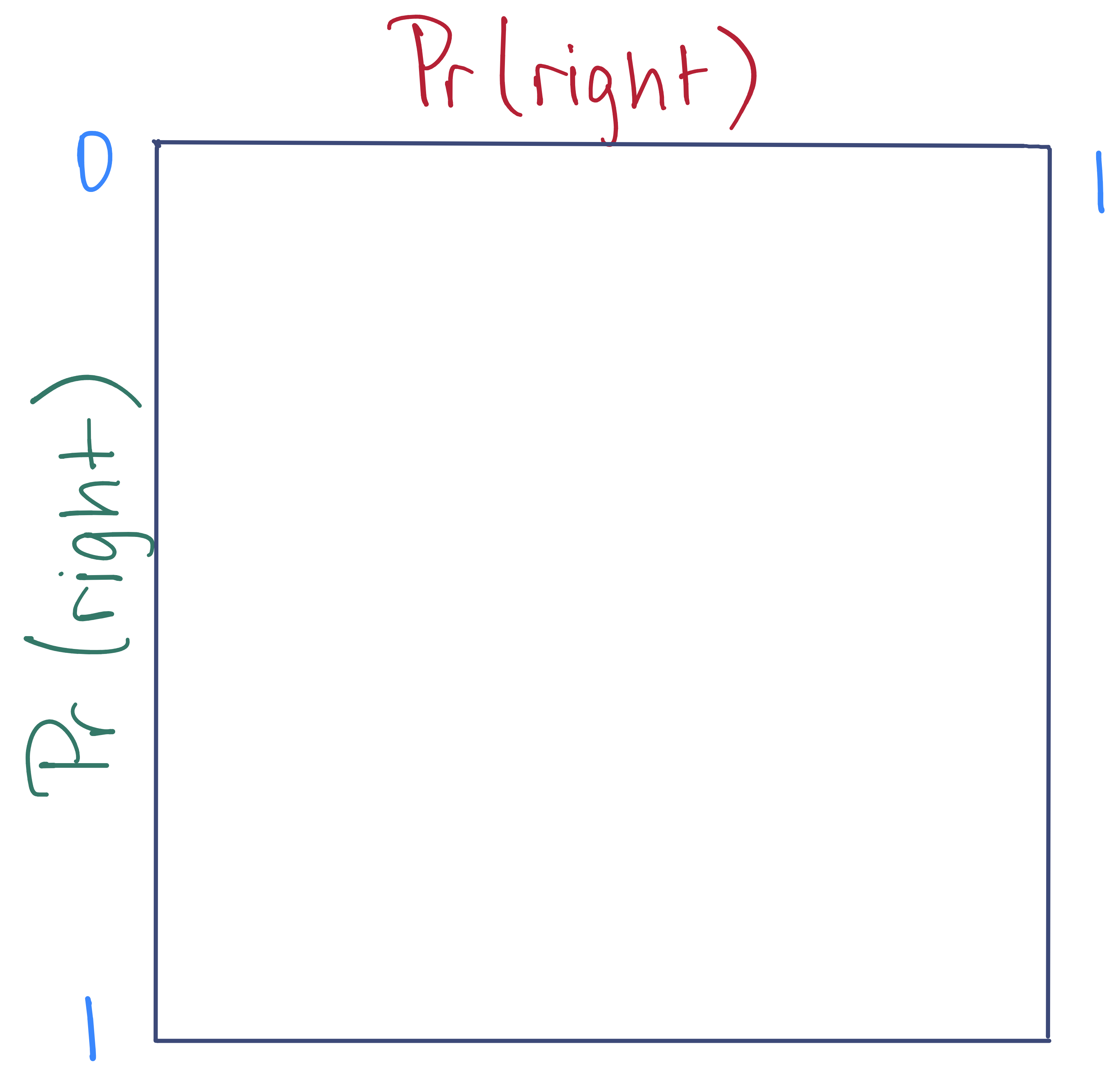
the fixed points correspond to NE.

Brouwer $\Rightarrow$ Nash.

DIVE \ KICK	Left	Right
Left	+1 −1	−1 −1
Right	−1 +1	+1 −1

Penalty Shot Game

$\longmapsto$



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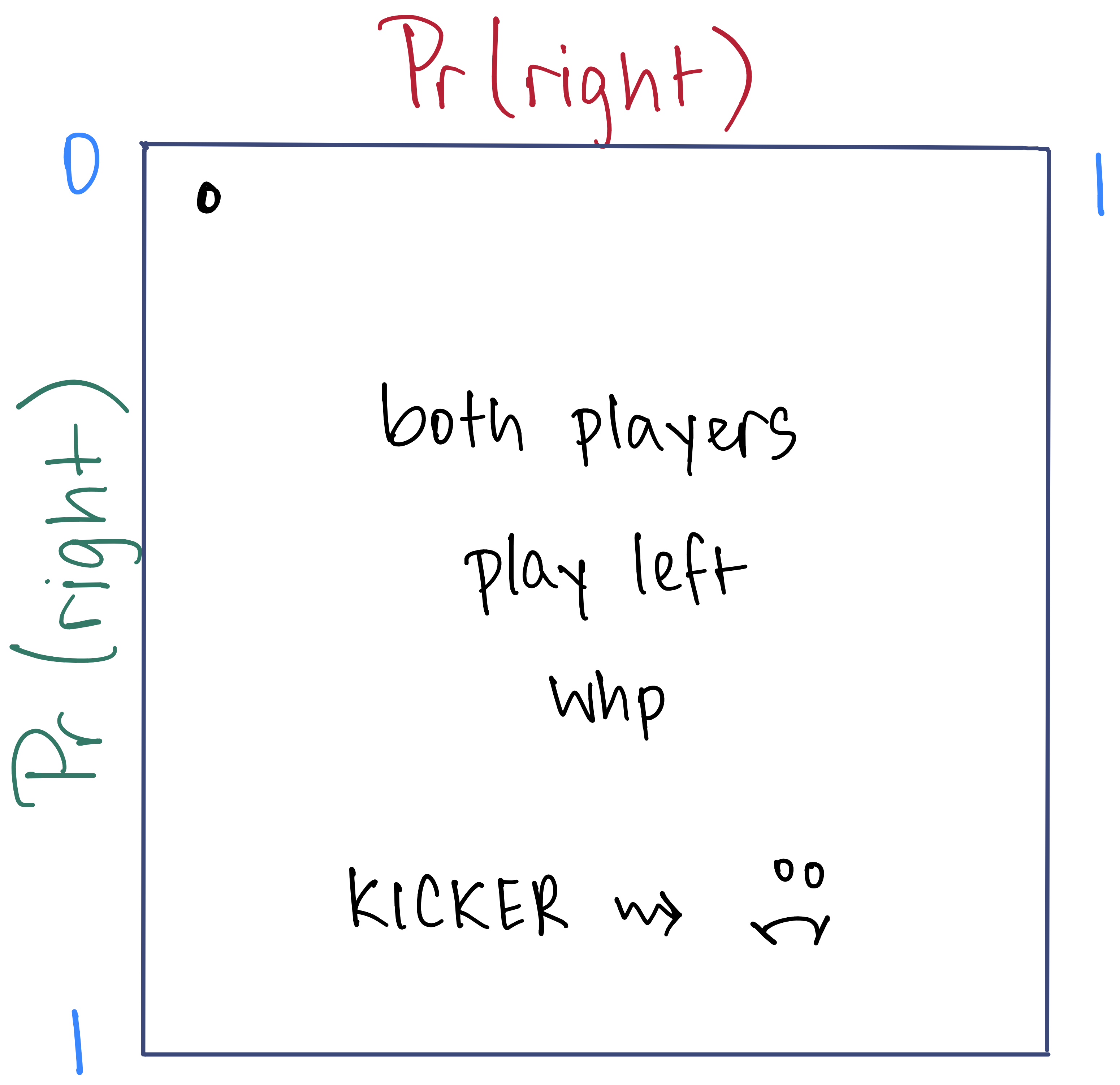
Penalty Shot Game



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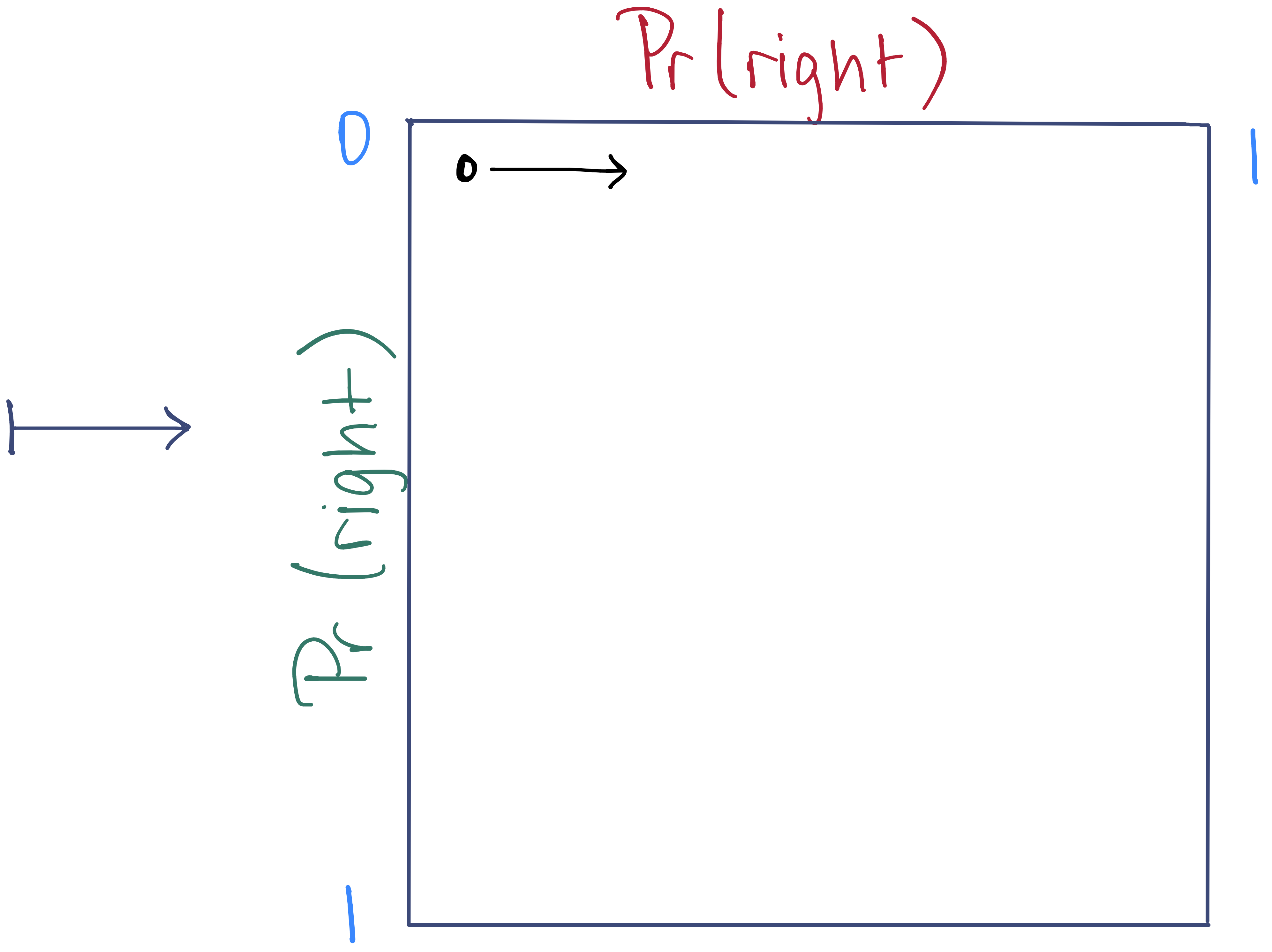
Penalty Shot Game

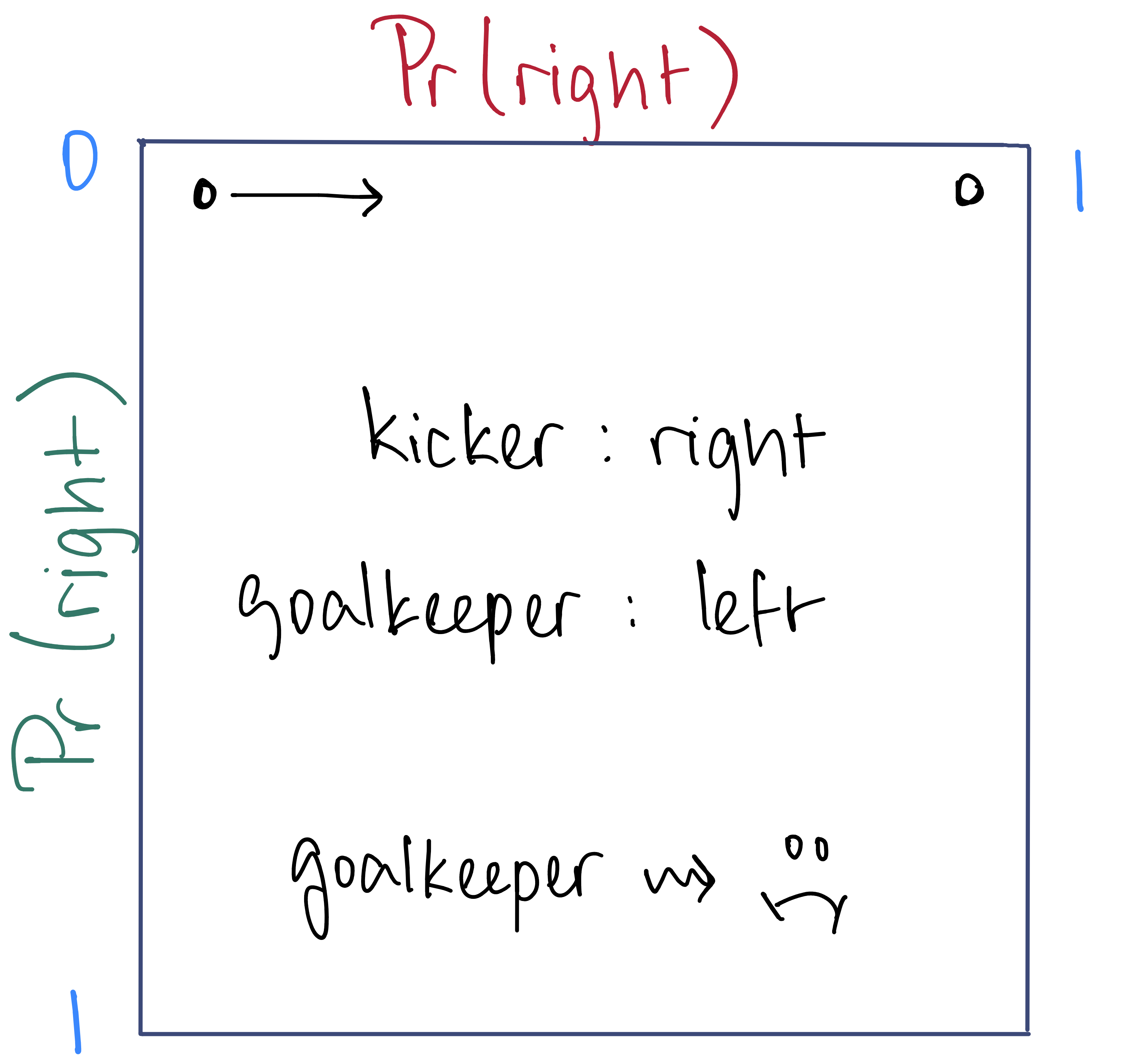


Brouwer $\Rightarrow$ Nash.

DIVE \ KICK	Left	Right
Left	$+1 \quad -1$	$-1 \quad -1$
Right	$-1 \quad +1$	$+1 \quad -1$

Penalty Shot Game

$\longmapsto$



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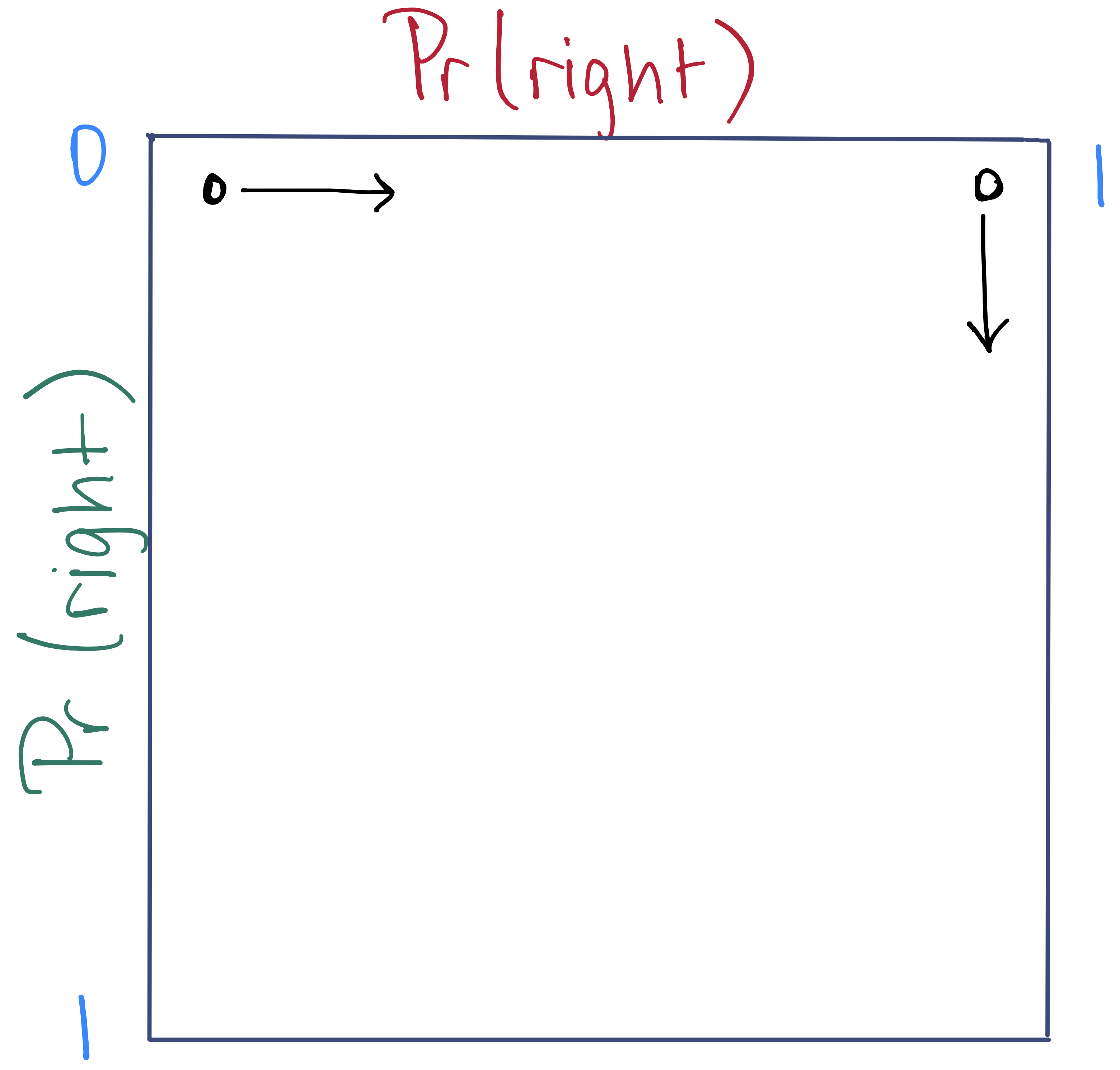

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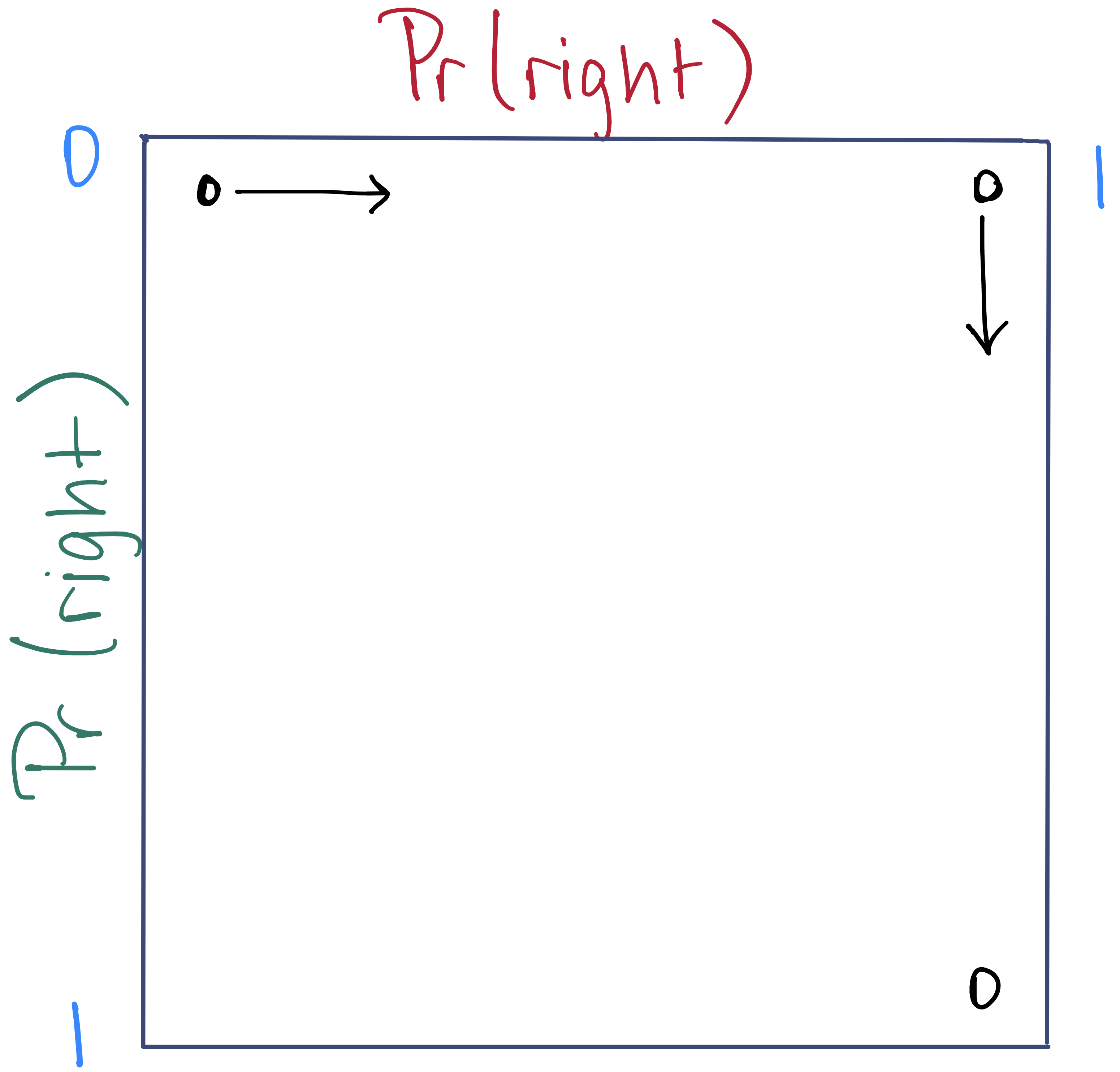

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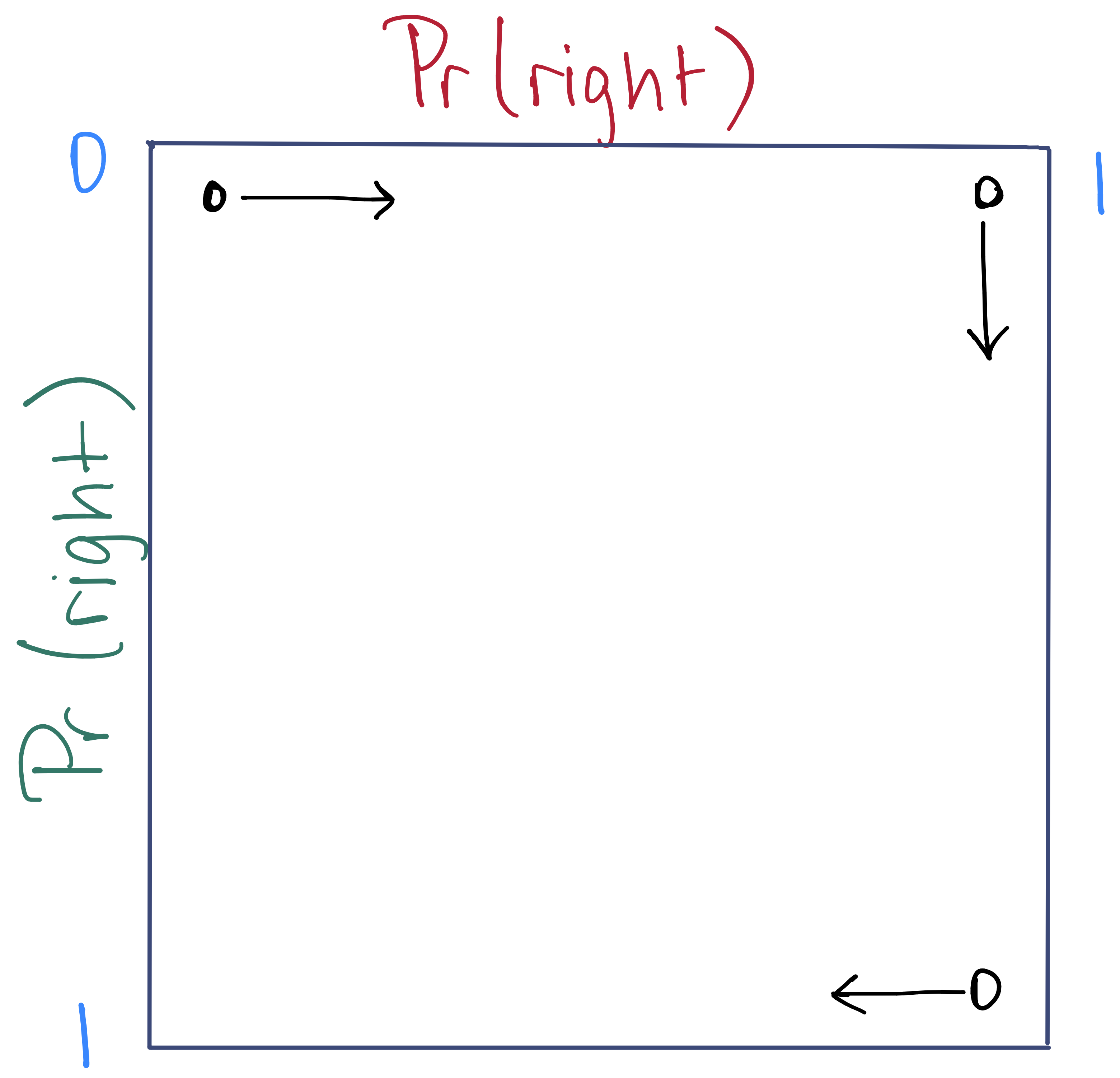

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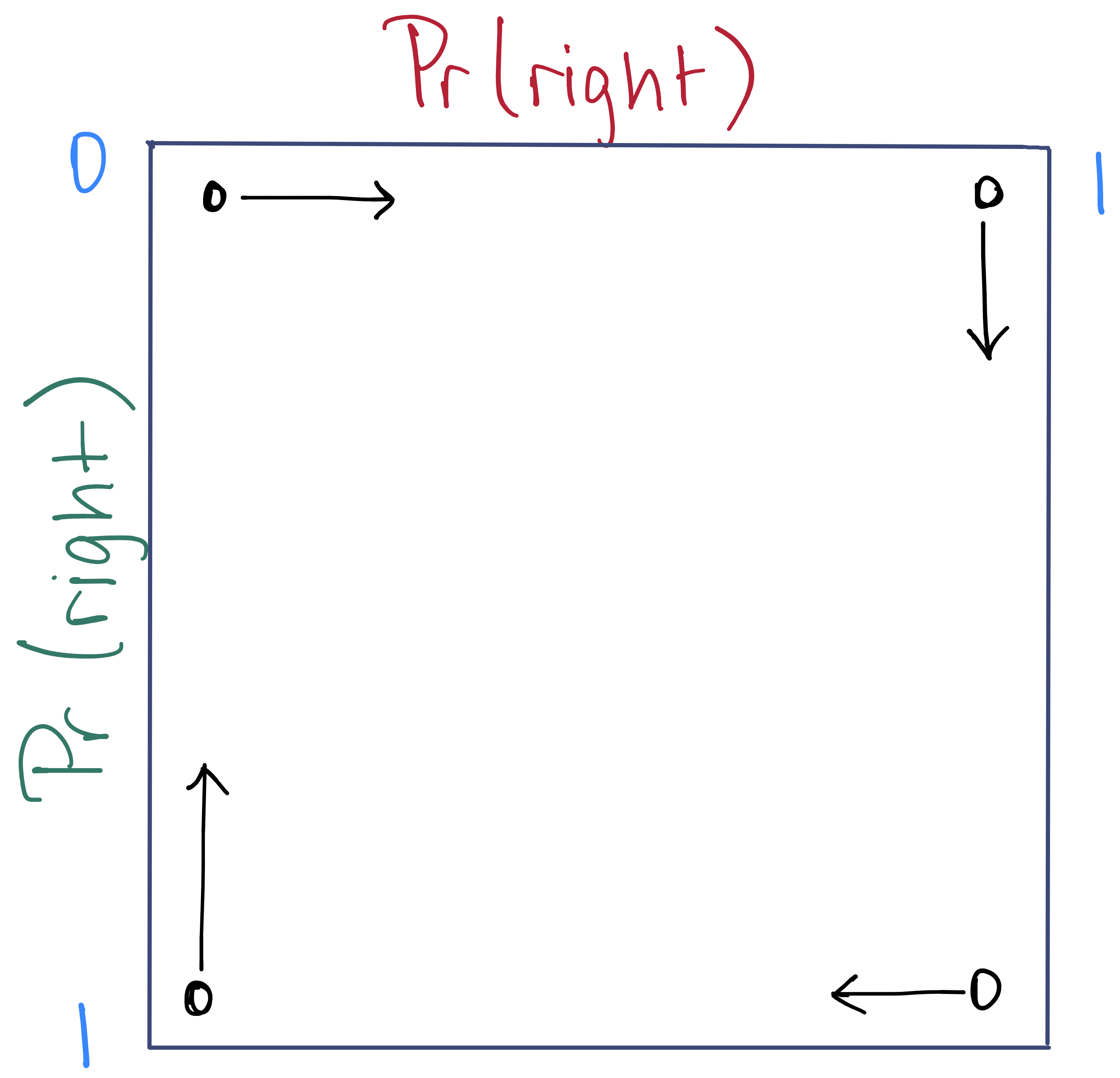

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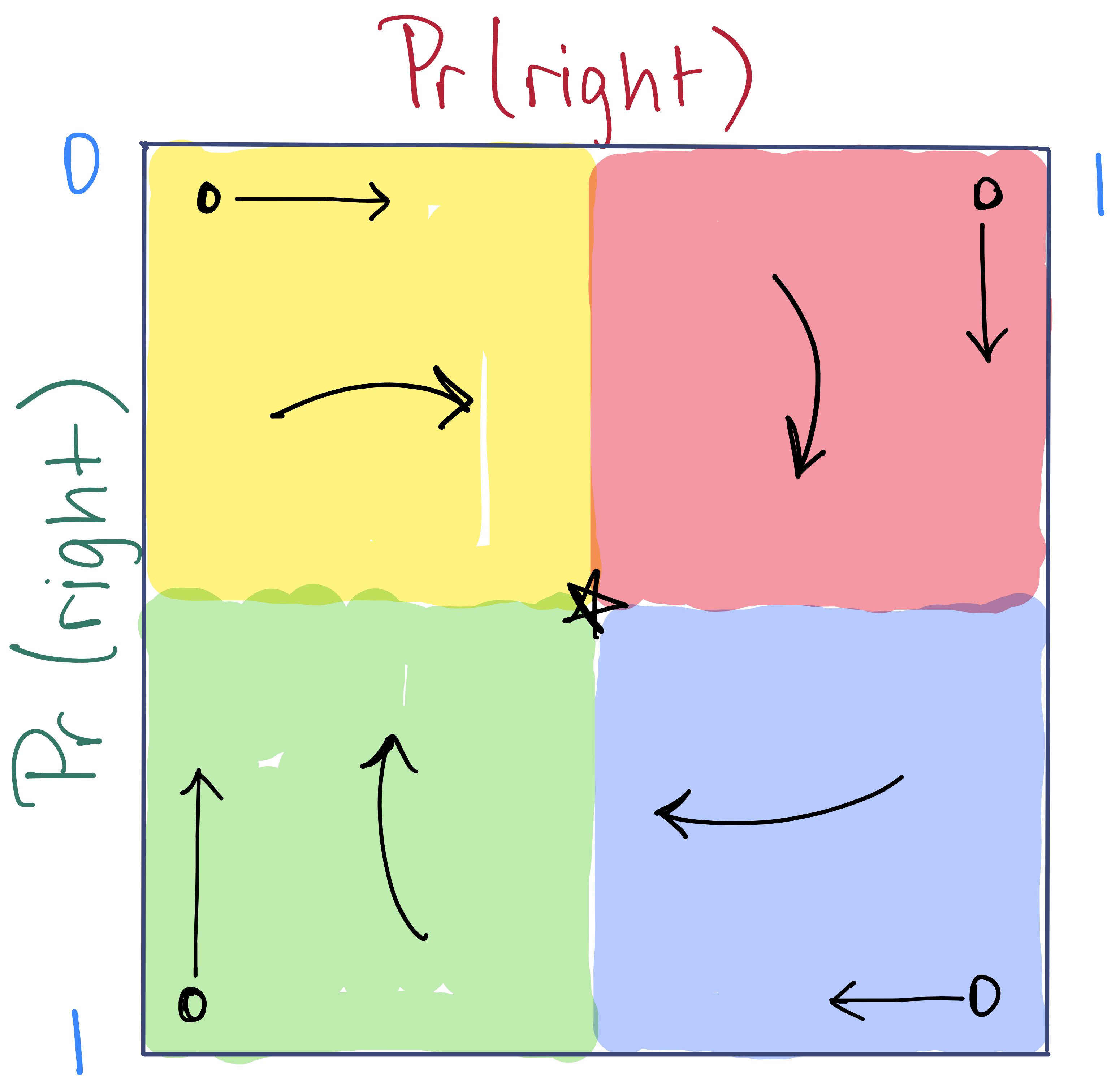
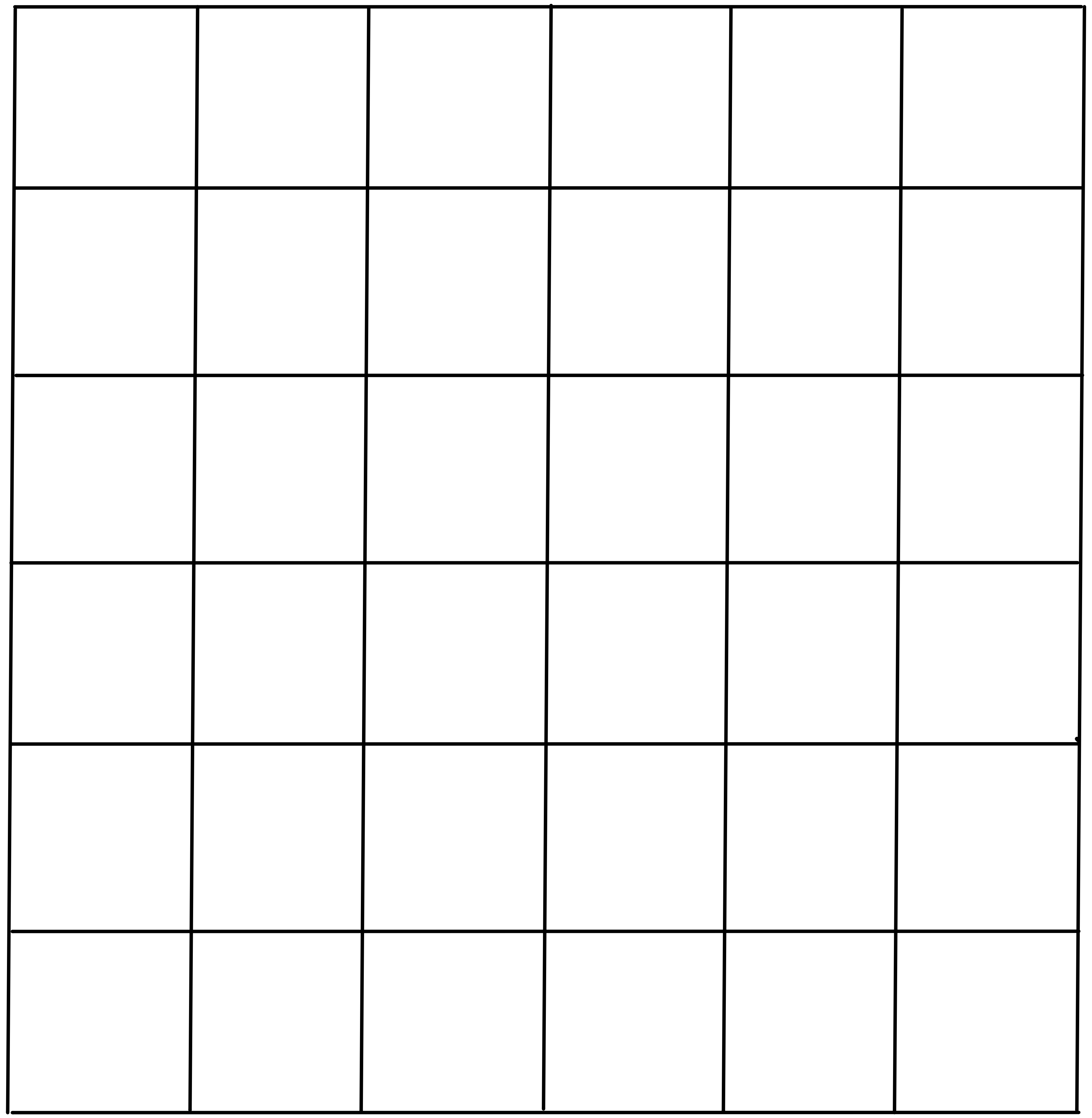


Sperner's Lemma (2D)



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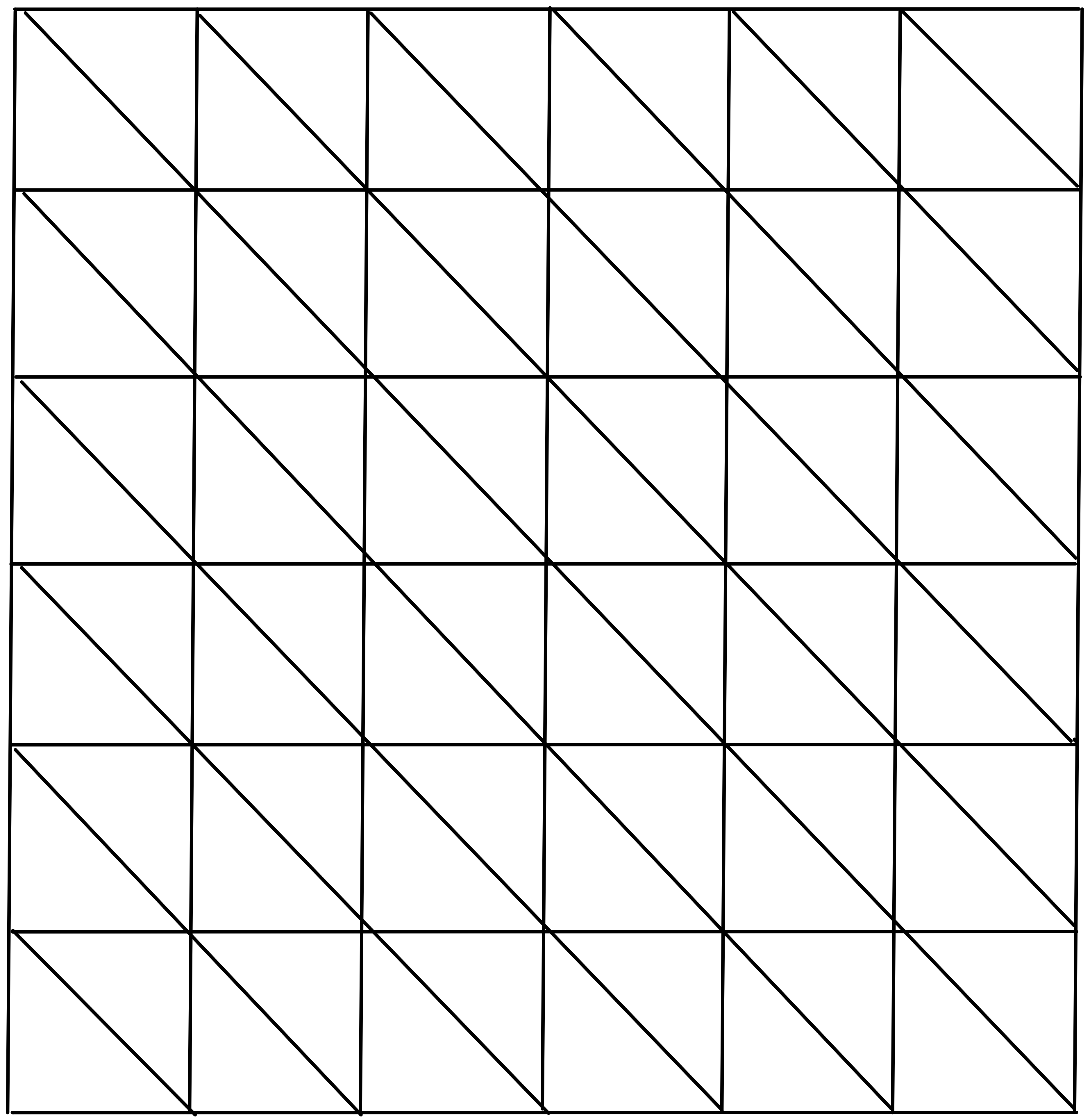


Sperner's Lemma (2D)

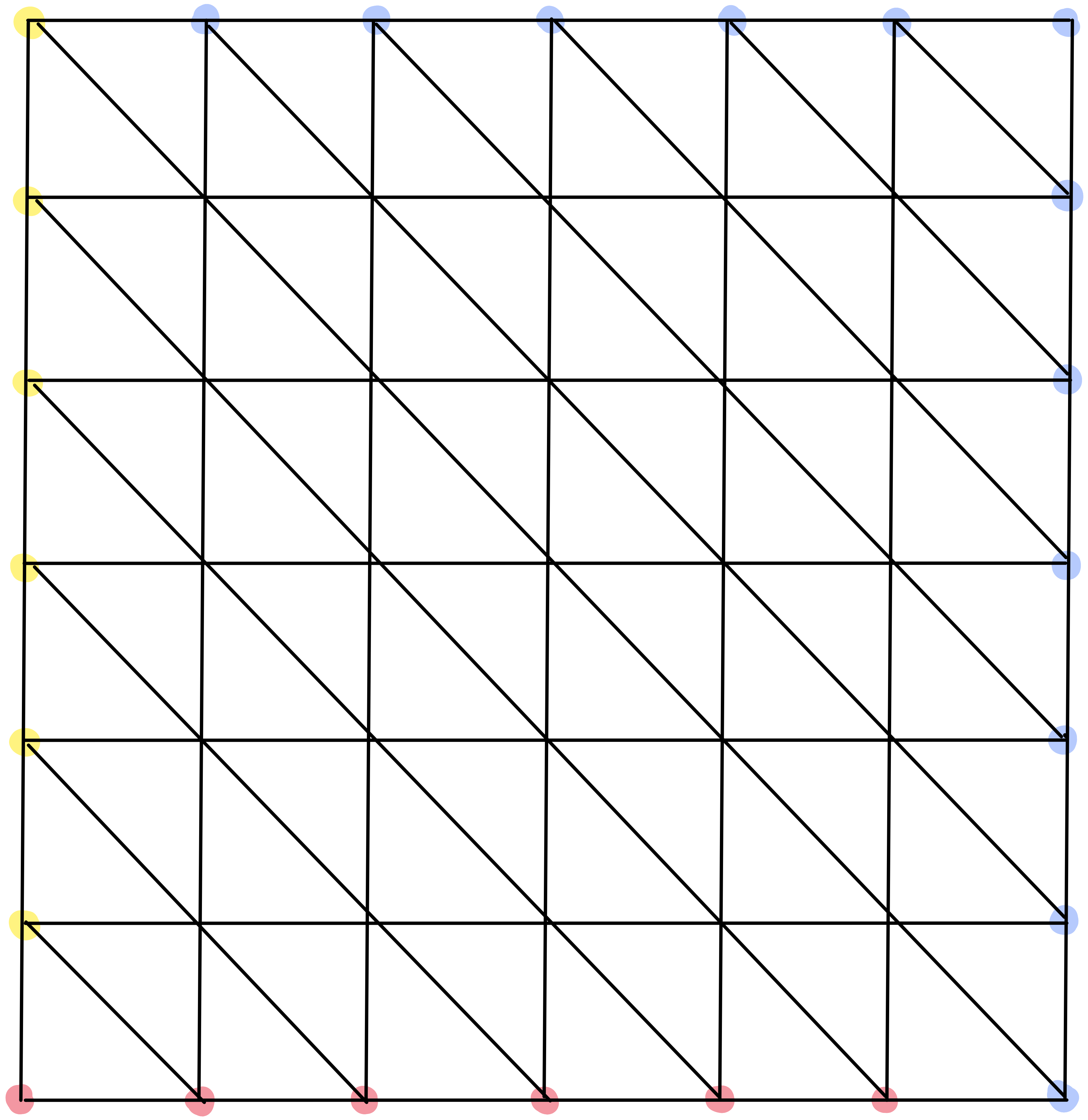
If

the boundary

is legally colored &

regardless of how the

internal nodes are colored



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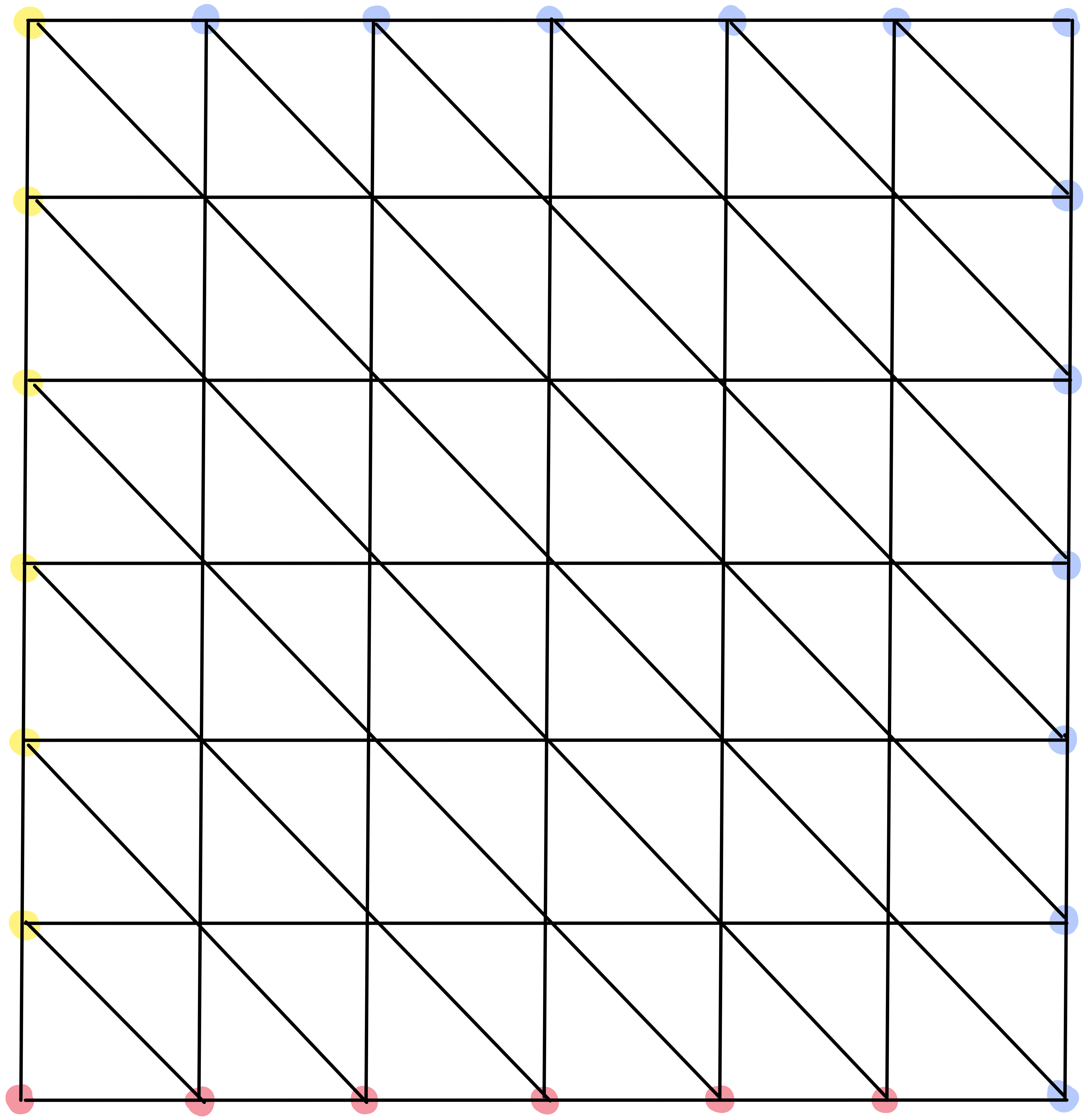

If

the boundary

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then

there exist an

odd # (& therefore

at least one) of

trichromatic Δ^{le}s.

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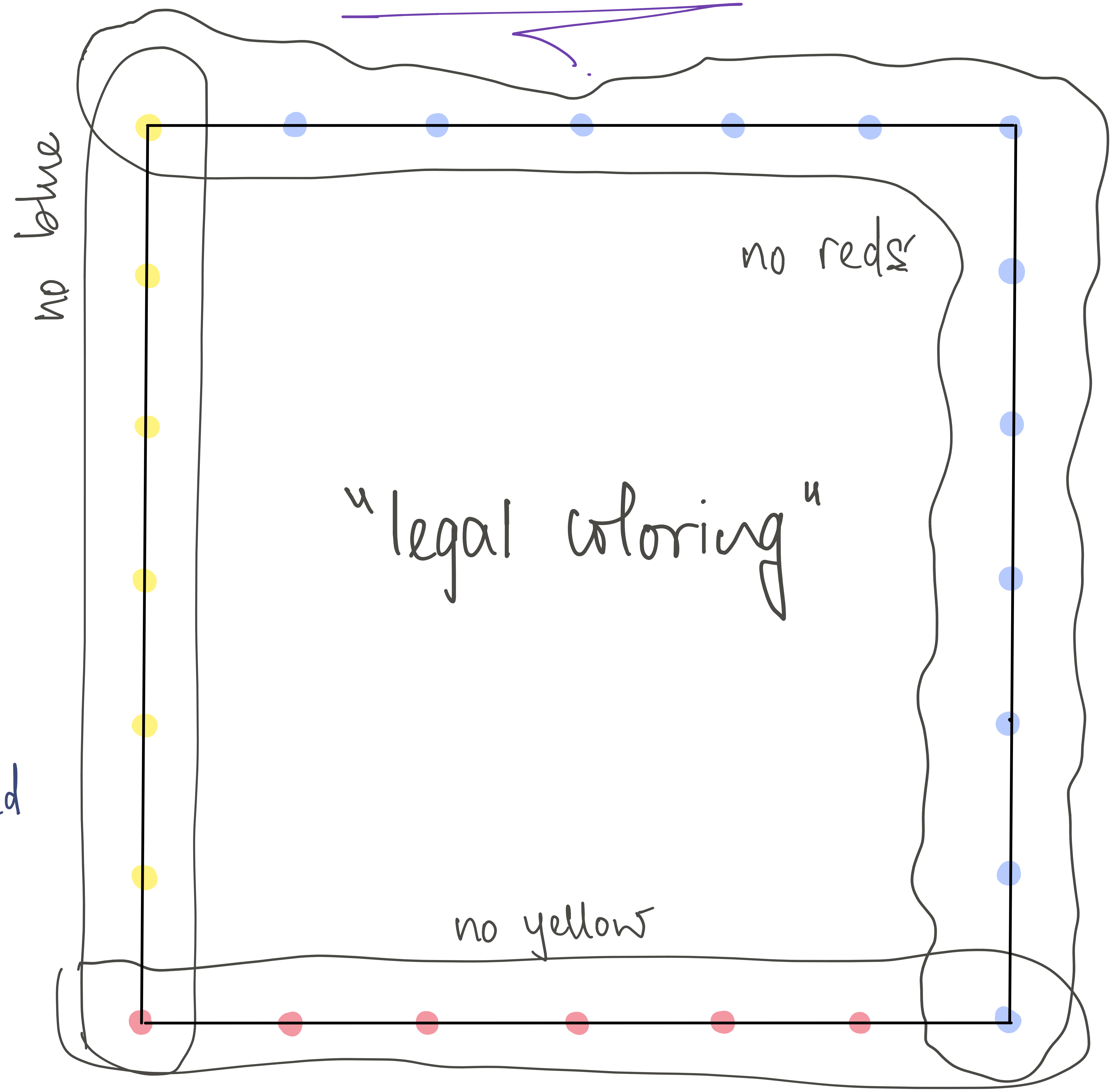

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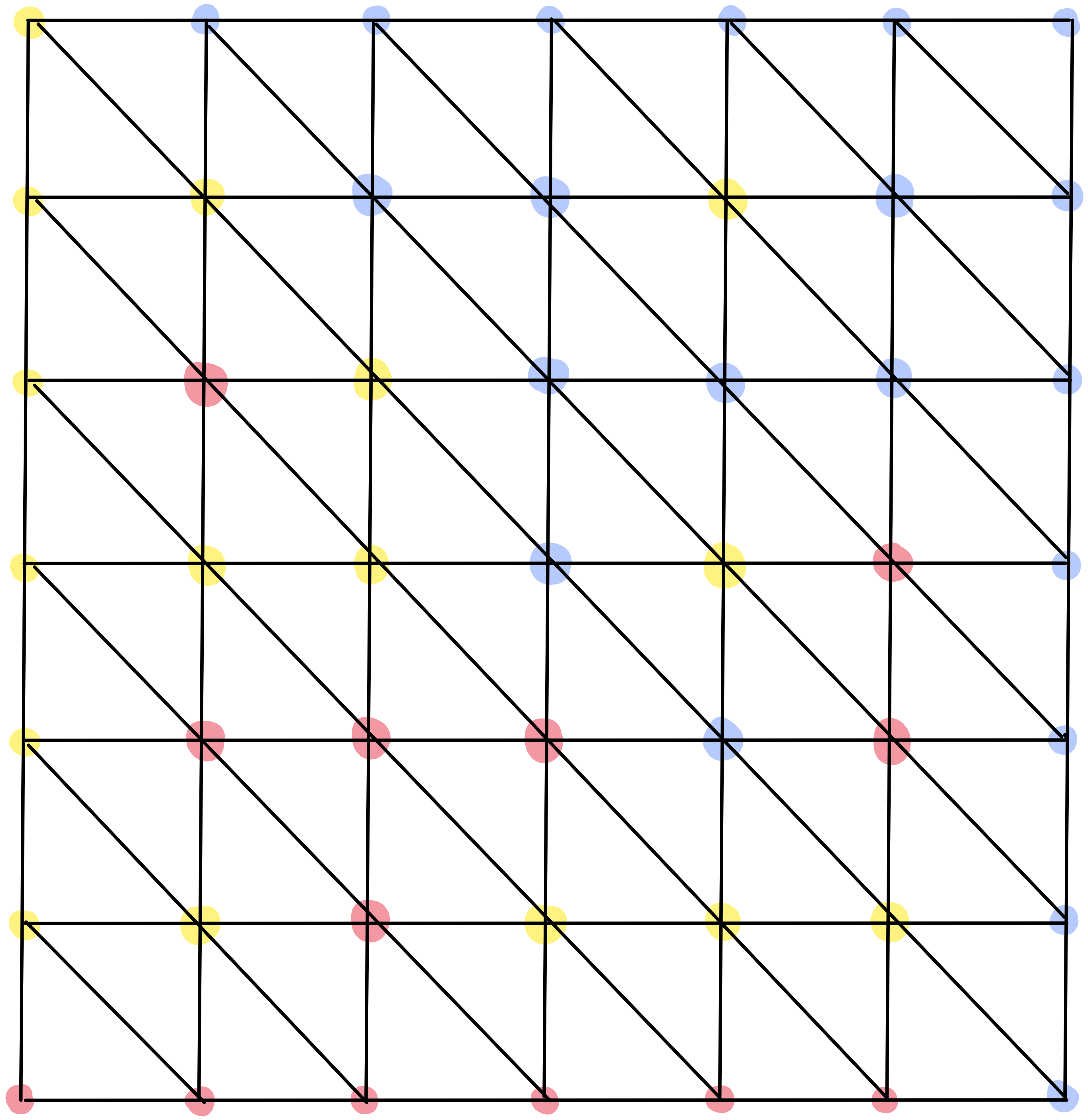

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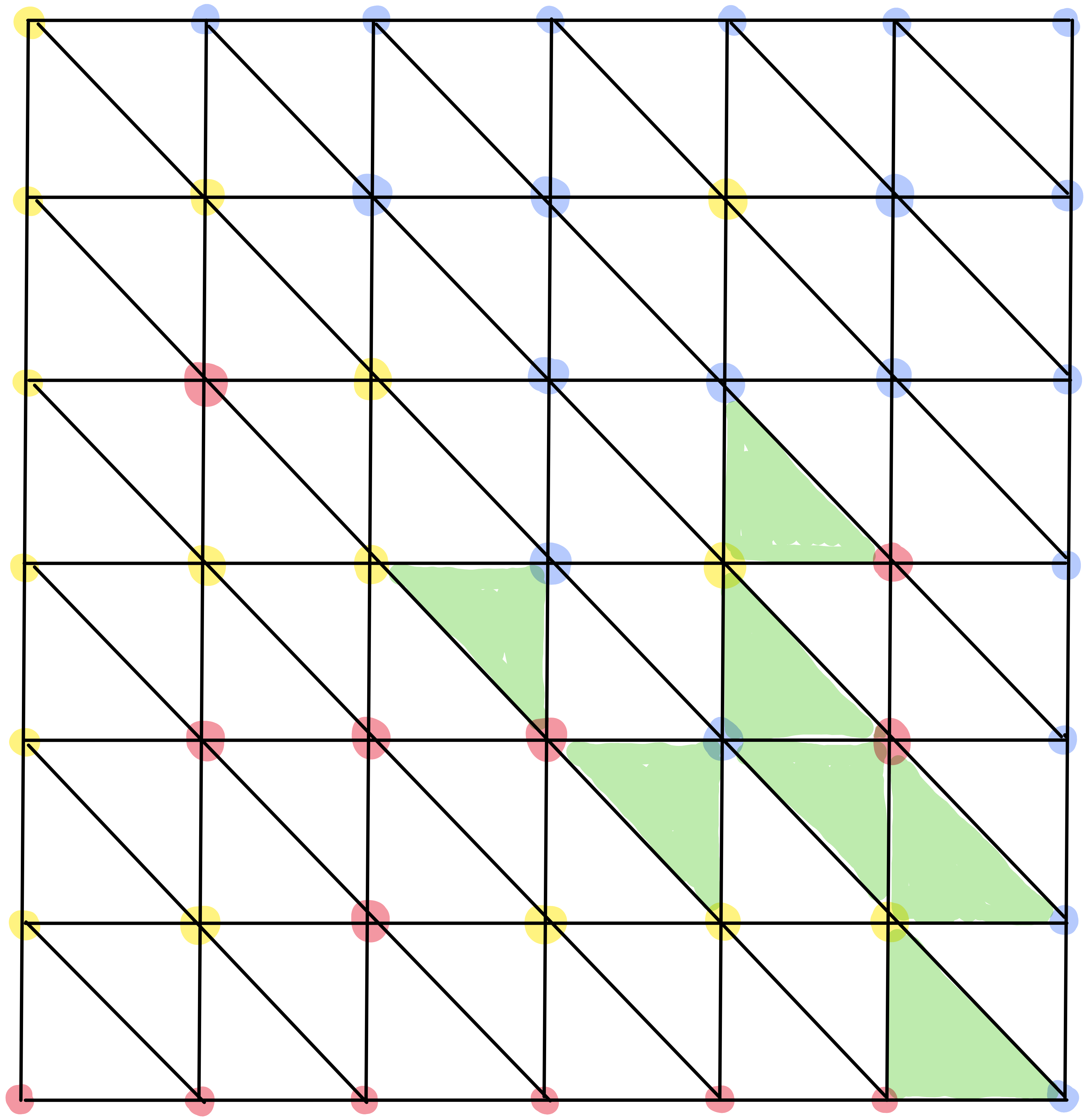

then

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trichromatic Δ'es.

Sperner $\Rightarrow$ Brouwer

$$f : [0,1]^2 \to [0,1]^2$$

1. For all ϵ, existence of an approximate fixed point

$$|f(x) - x| < \epsilon$$

can be shown using Sperner

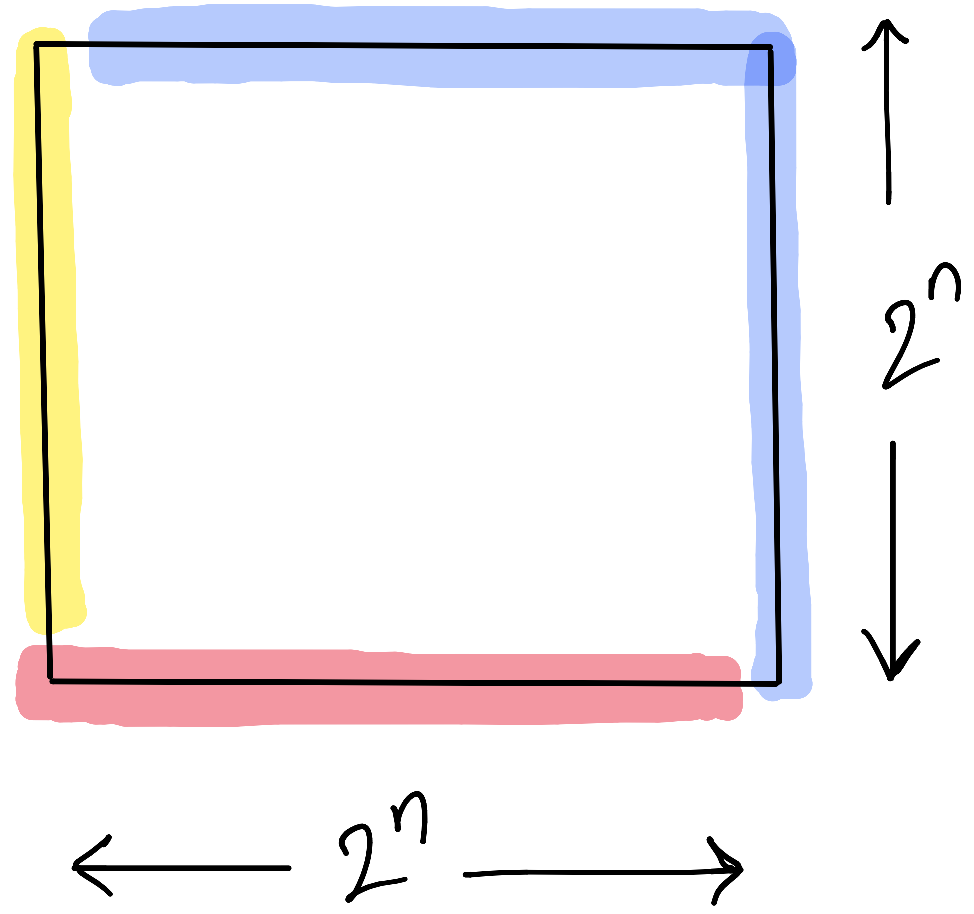
2. Then use compactness.

Sperner ⤳ The Computational Edition

Input : Grid of size 2^n with the standard coloring on the outer boundary

Colorings of the internal vertices are given by an oracle :

$$col(x,y) \to \{ \bullet , \bullet , \bullet \}$$



Output : A trichromatic Δ^{le}.

Nash Equilibrium ⇝ The Computational Version

Input. ① A game given by

- ⋆ # of players (n)
- ⋆ Strategy sets S_p for all $p \in 1, 2, \ldots, n$
- ⋆ Utility function $u_p : S \to \mathbb{R}$ for all p

② An approximation requirement ϵ.

Nash Equilibrium ⇝ The Computational Version

Output. An ϵ-Nash Equilibrium of the game *

$$\forall p. \quad U_p(x_1, \ldots, x_n) \geq U_p(x_1, \ldots, x'_p, \ldots, x_n) - \epsilon$$

$$\forall x'_p \in \Delta(S_p)$$

* helps guarantee polynomial output complexity.

Function NP (FNP)

(capturing search problems)

Search Problem. L is defined by a relation

$$R_L \subseteq \{0,1\}^* \times \{0,1\}^*$$

$(x,y) \in R_L$ iff y is a solution to x.

Function NP (FNP)

(capturing search problems)

Total Search Problem. L is defined by a relation

$$R_L \subseteq \{0,1\}^* \times \{0,1\}^*$$

$(x,y) \in R_L$ iff y is a solution to x.

L is total iff $\forall x \; \exists y$ s.t. $(x,y) \in R_L$.

Function NP (FNP)

(capturing search problems)

A search problem is in FNP iff

$\exists$ a poly time algorithm $A_L(\cdot,\cdot)$ & a polynomial function $p_L(\cdot)$ s.t.

Function NP (FNP)

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$\forall x$. $\exists y$ s.t. $(x,y) \in R_L \Rightarrow \exists z$ with $|z| \leq p_L(|x|)$ s.t. $(x,z) \in R_L$

Function NP (FNP)

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$\forall x. \quad \exists y$ s.t. $(x,y) \in R_L \Rightarrow \exists z$ with $|z| \le p_L(|x|)$ s.t. $(x,z) \in R_L$

$$TFNP = \{ L \in FNP \mid L \text{ is total} \}.$$

$$\left.\begin{array}{l}\text{Sperner}\\ \text{Nash}\\ \text{Brouwer}\end{array}\right\} \in \text{TFNP}$$

$$L \in FNP \xrightarrow[\text{Reduction}]{\text{Karp}} L' \in FNP$$

$$(A_L, P_L) \qquad f \qquad (A_{L'}, P_{L'})$$

$f : \{0,1\}^* \to \{0,1\}^*$ maps inputs of L to inputs to L'.

$$\forall x, y : \quad A_{L'}(f(x), y) = 1 \quad \Rightarrow \quad A_L(x, g(y)) = 1$$

if $f(x)$ has a solution under L' then x must have a solution under L

$$\forall x : \quad A_{L'}(f(x), y) = 0 \ \forall y \quad \Rightarrow \quad A_L(x, y) = 0 \ \forall y$$

if $f(x)$ has no solution then x has no solution either.

A search problem is FNP-complete if

$L \in FNP$

& $\forall L' \in FNP$, L' is poly-time reducible to L.

(SAT is FNP-complete.)

$L \in FNP$ $\xrightarrow[\text{Reduction}]{\text{Karp}}$ $L' \in FNP$

SAT (A_L, P_L) f $(A_{L'}, P_{L'})$ Sperner

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A Complexity Theory

of

Total Search Problems

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1 identify a combinatorial argument of existence that makes these search problems total

A Complexity Theory of Total Search Problems

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2 define a complexity class "inspired" by the argument of existence

A Complexity Theory of Total Search Problems

1. identify a combinatorial argument of existence that makes these search problems total

2. define a complexity class "inspired" by the argument of existence

3. Justify the definition

:)

Sperner's Lemma (2D)

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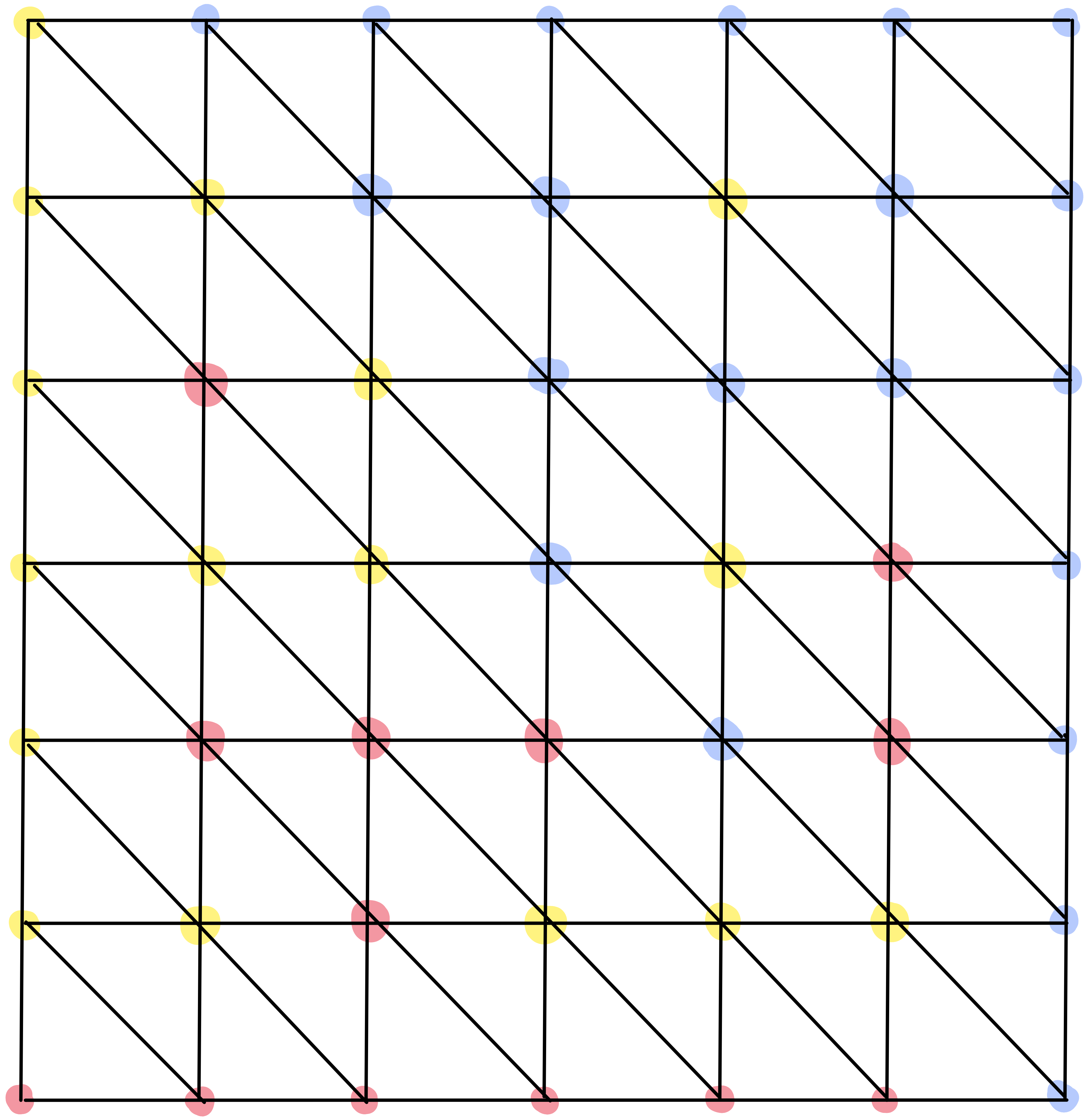

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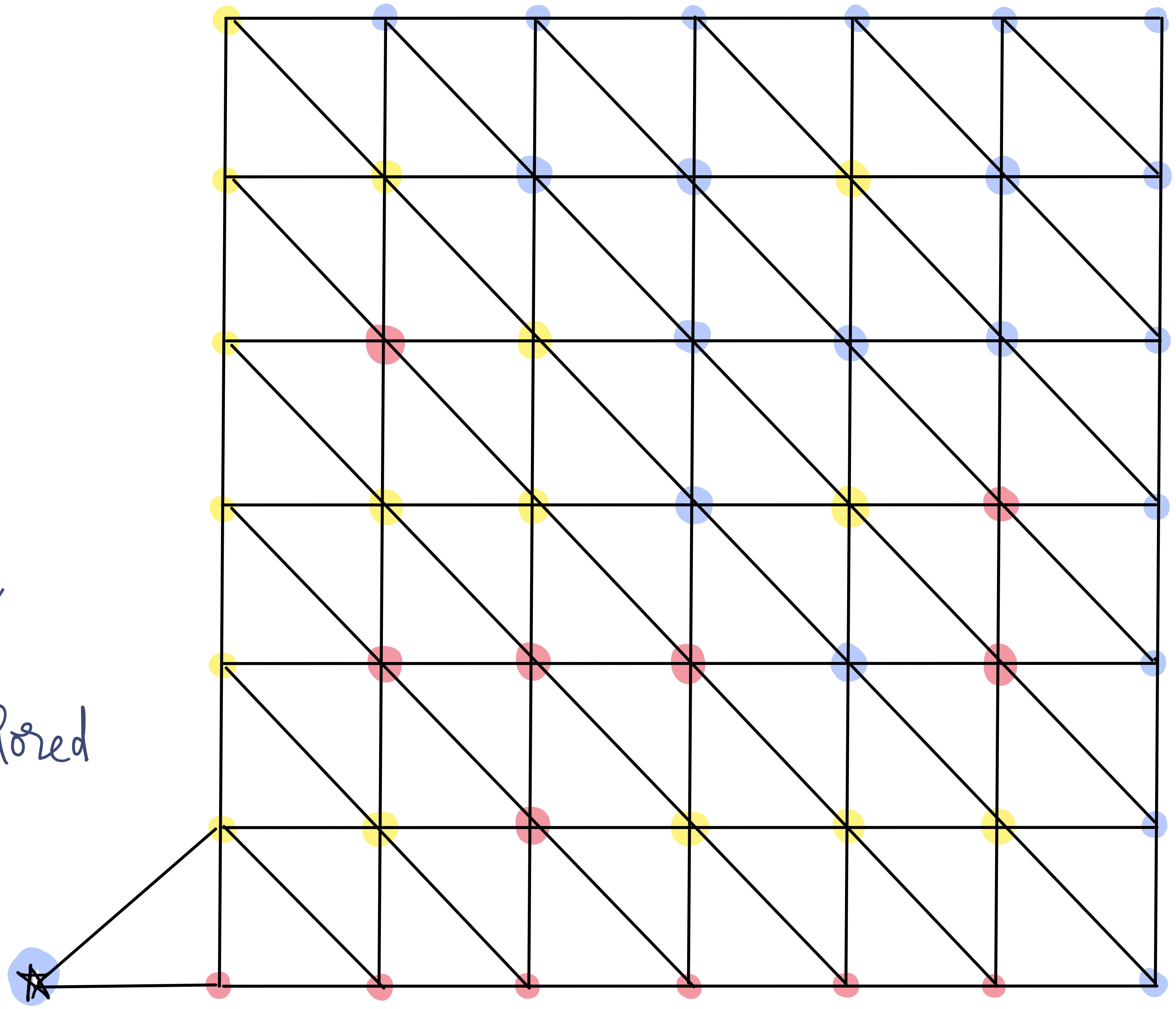

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Sperner's Lemma (2D)

only pursue
R-Y doors
w/ Y on your
left

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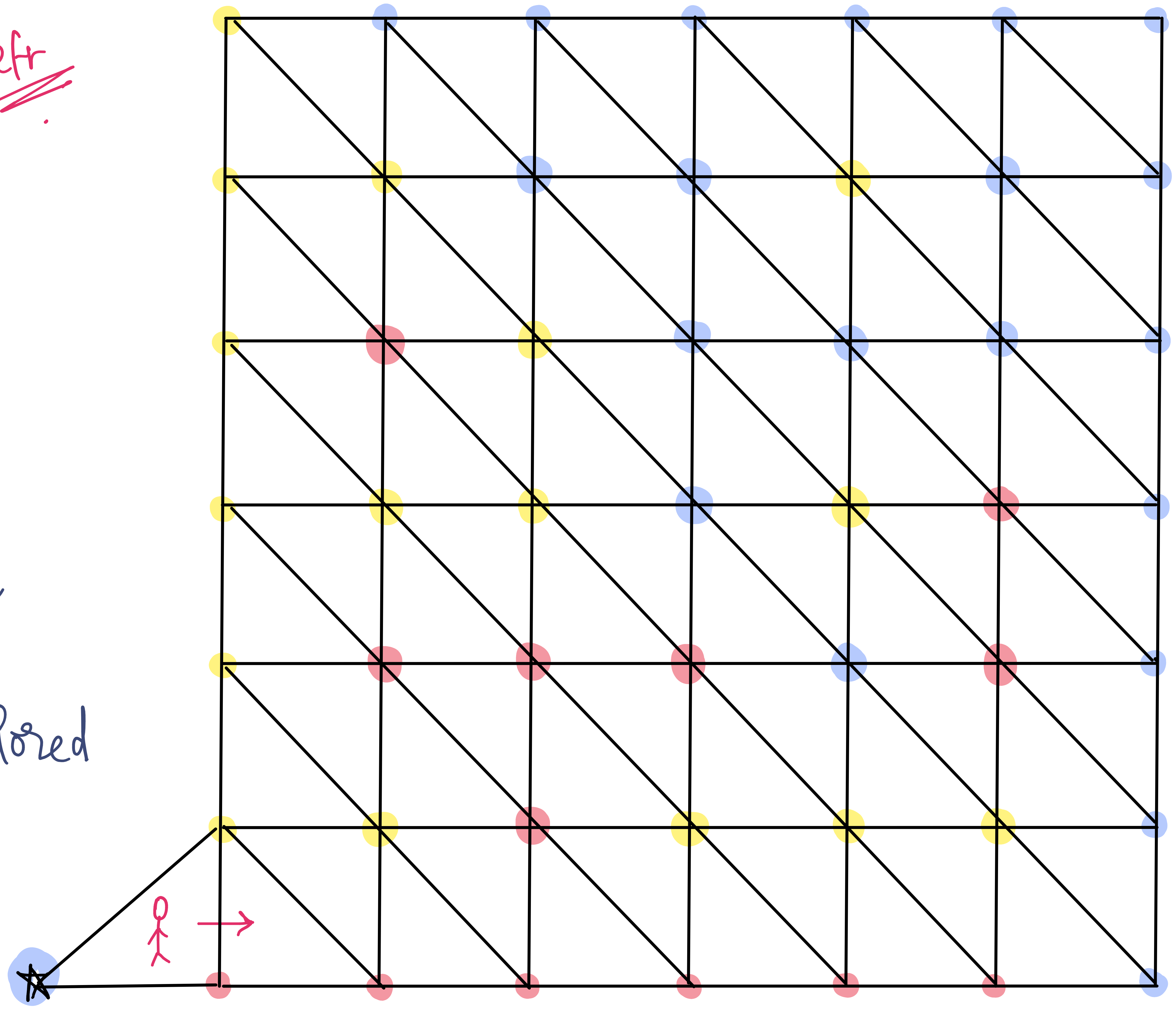

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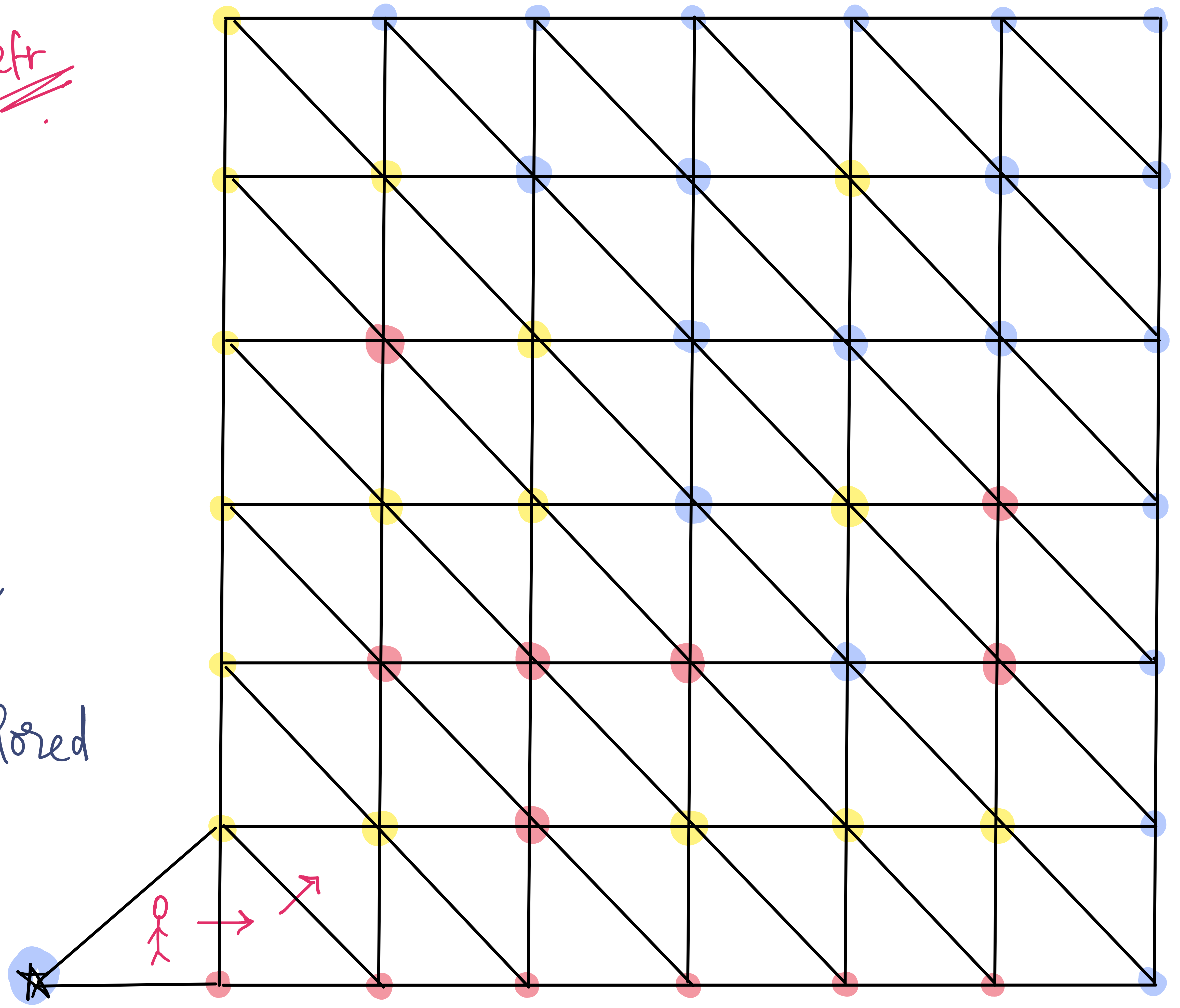

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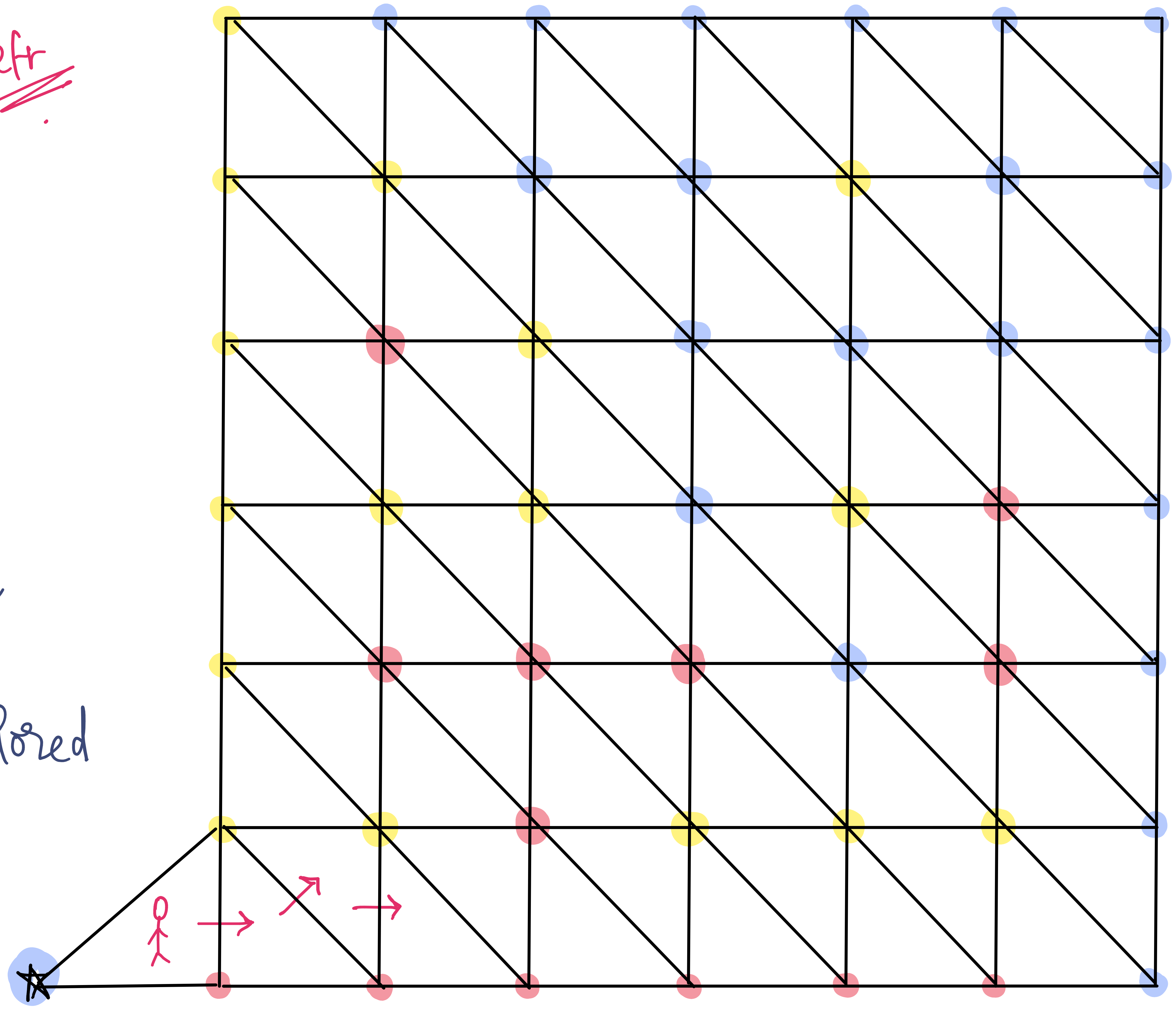

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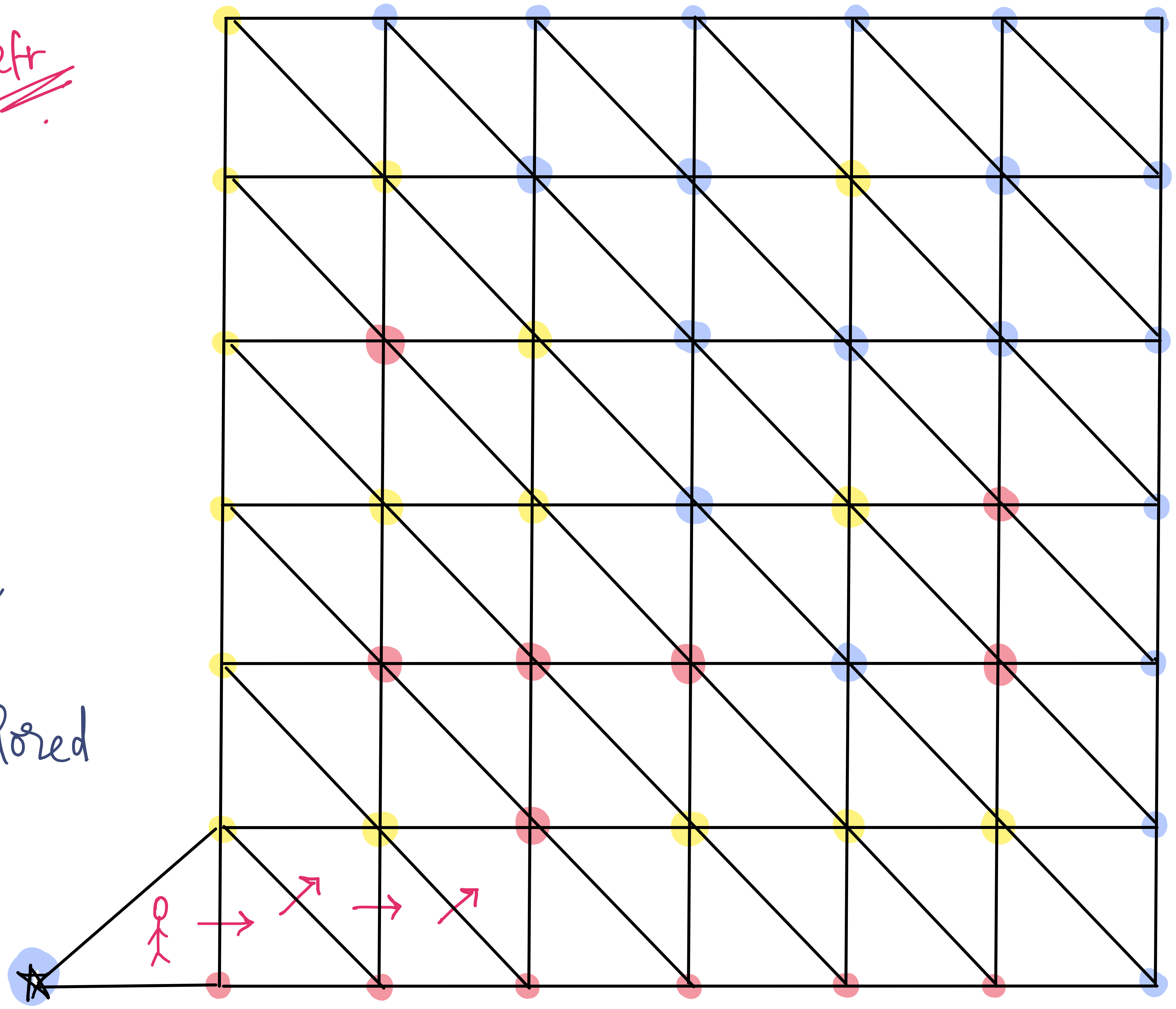
only pursue R-Y doors w/ Y on your left

If the boundary is legally colored & regardless of how the internal nodes are colored



then there exist an odd # (& therefore at least one) of trichromatic Δ's.

Sperner's Lemma (2D)

only pursue
R-Y doors
w/ Y on your
~~left~~

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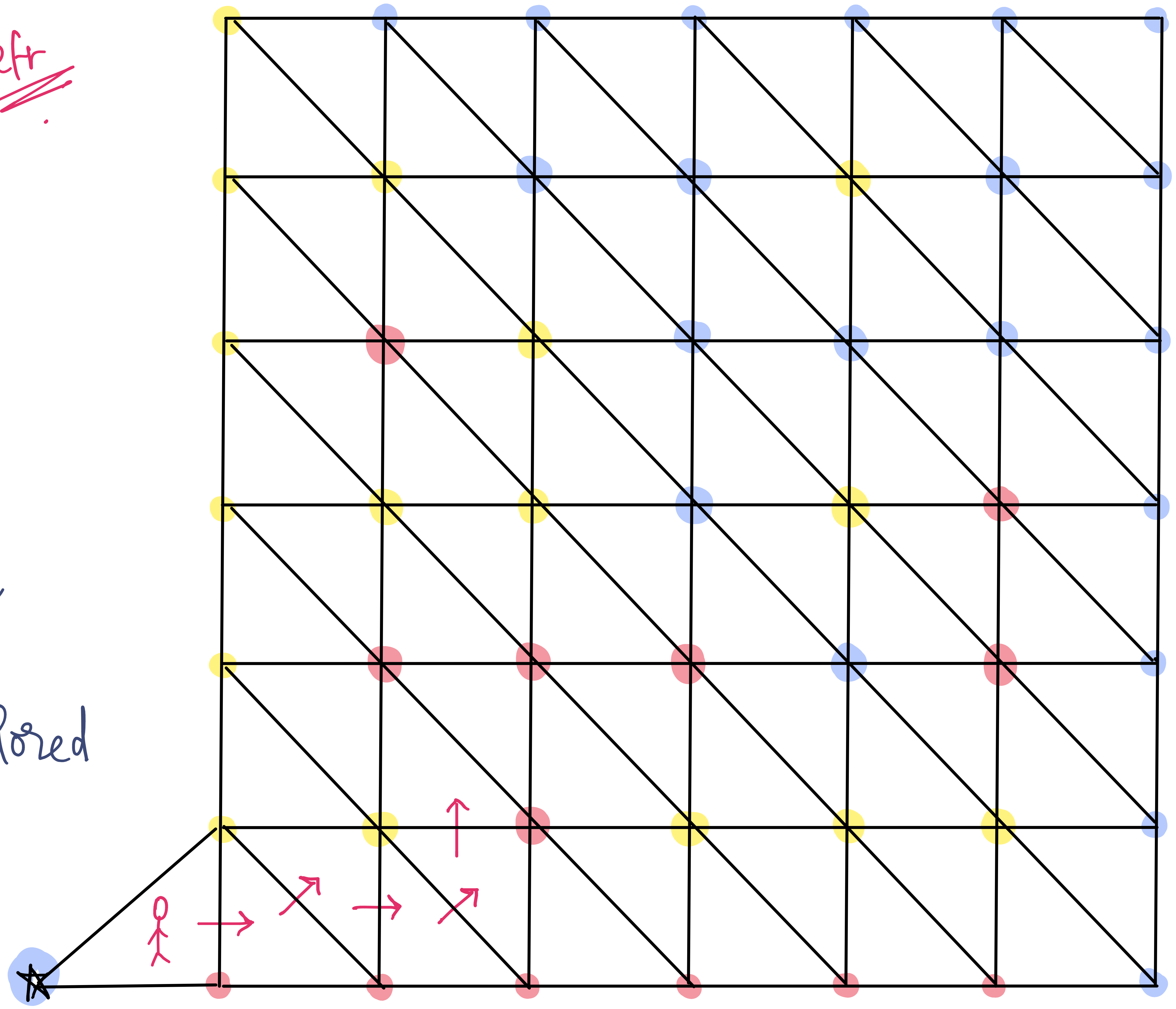

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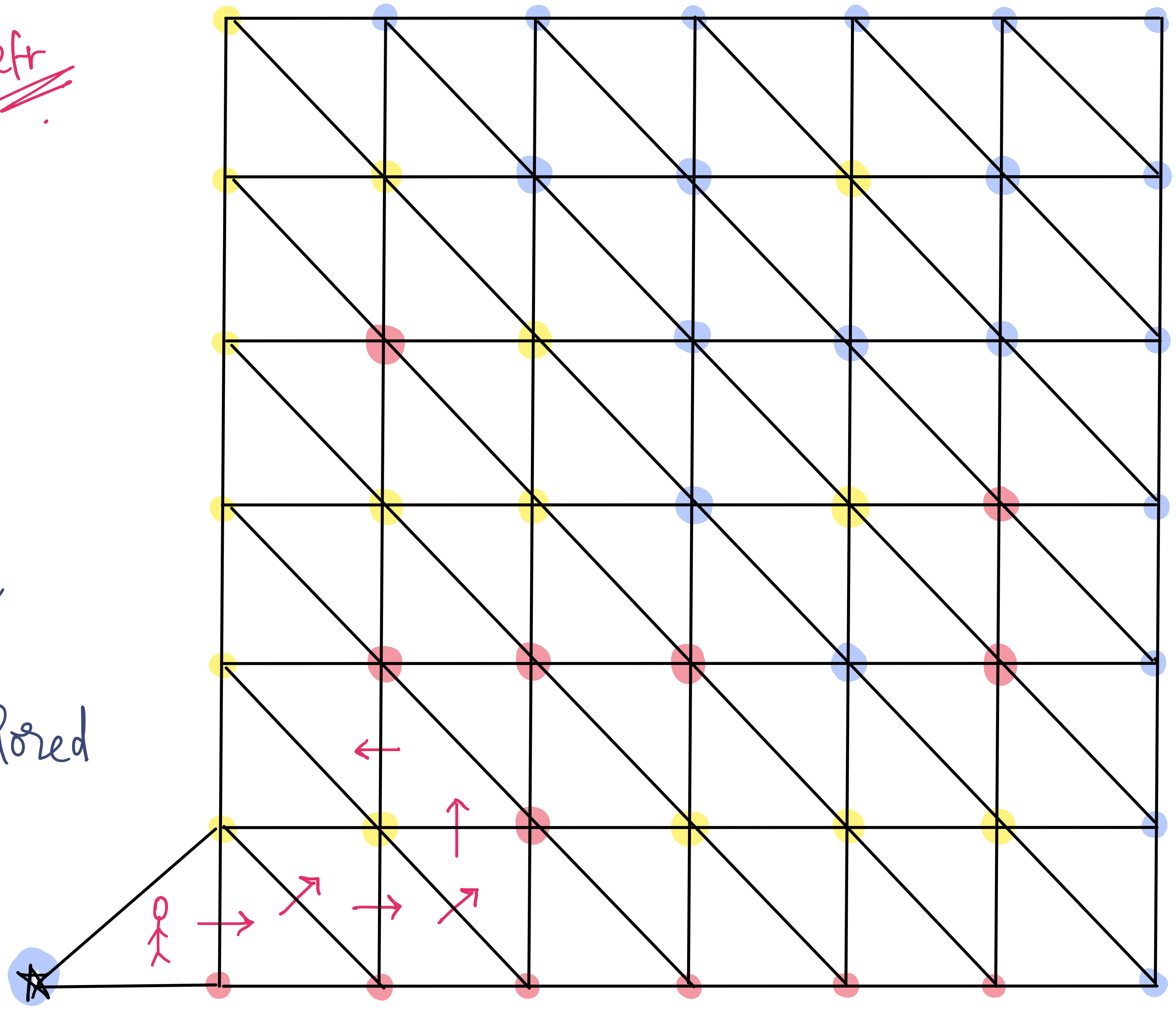

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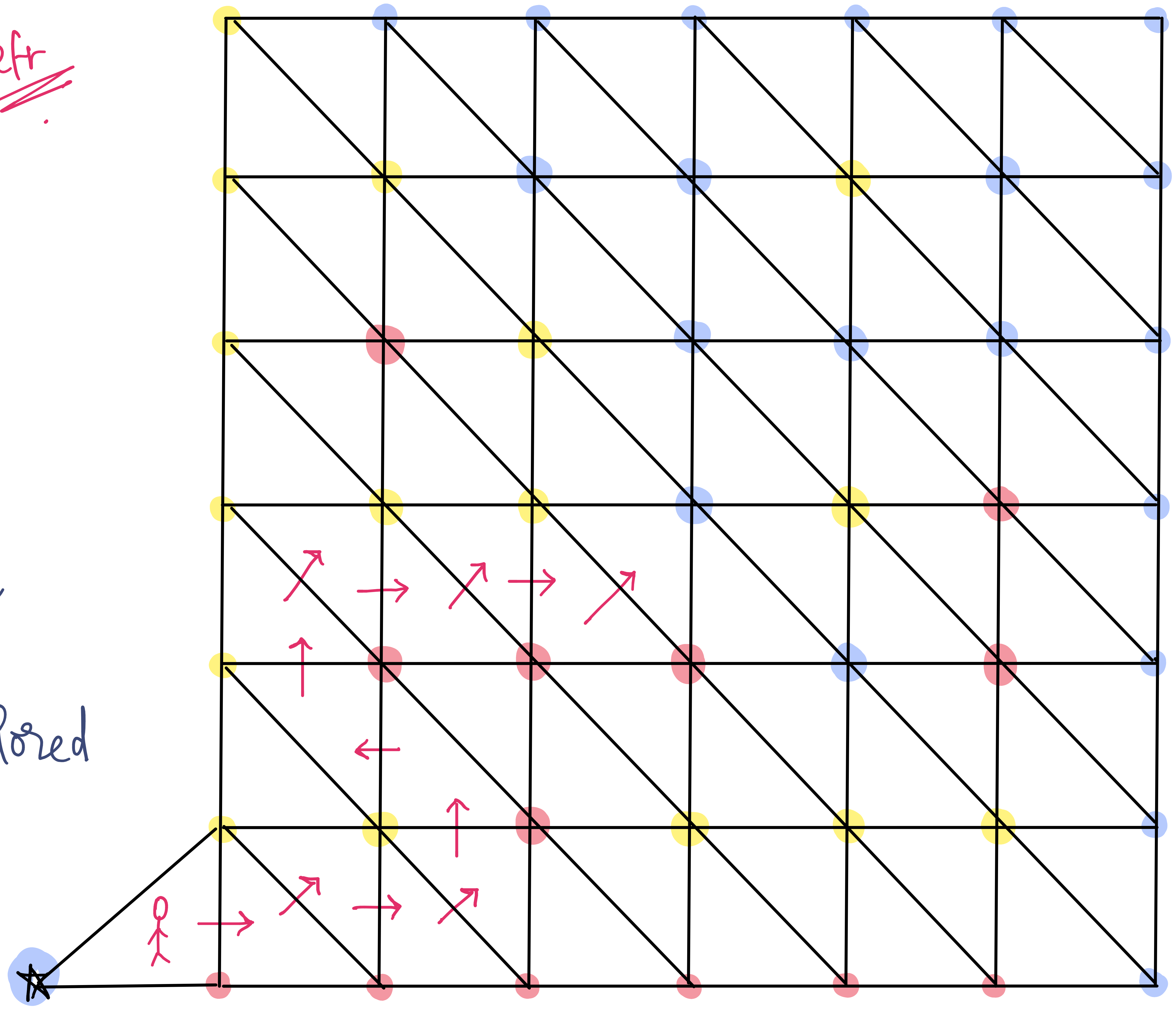

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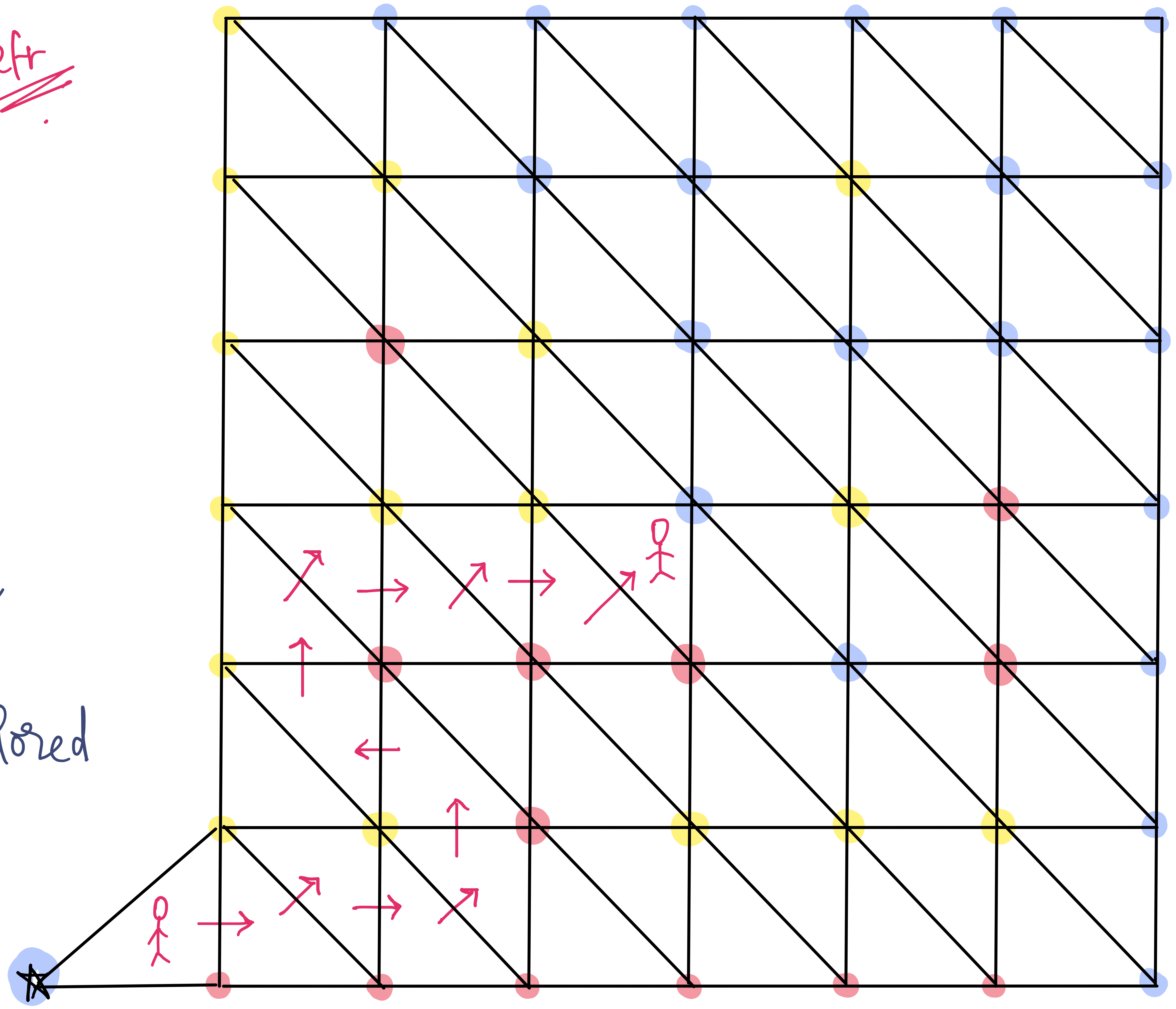

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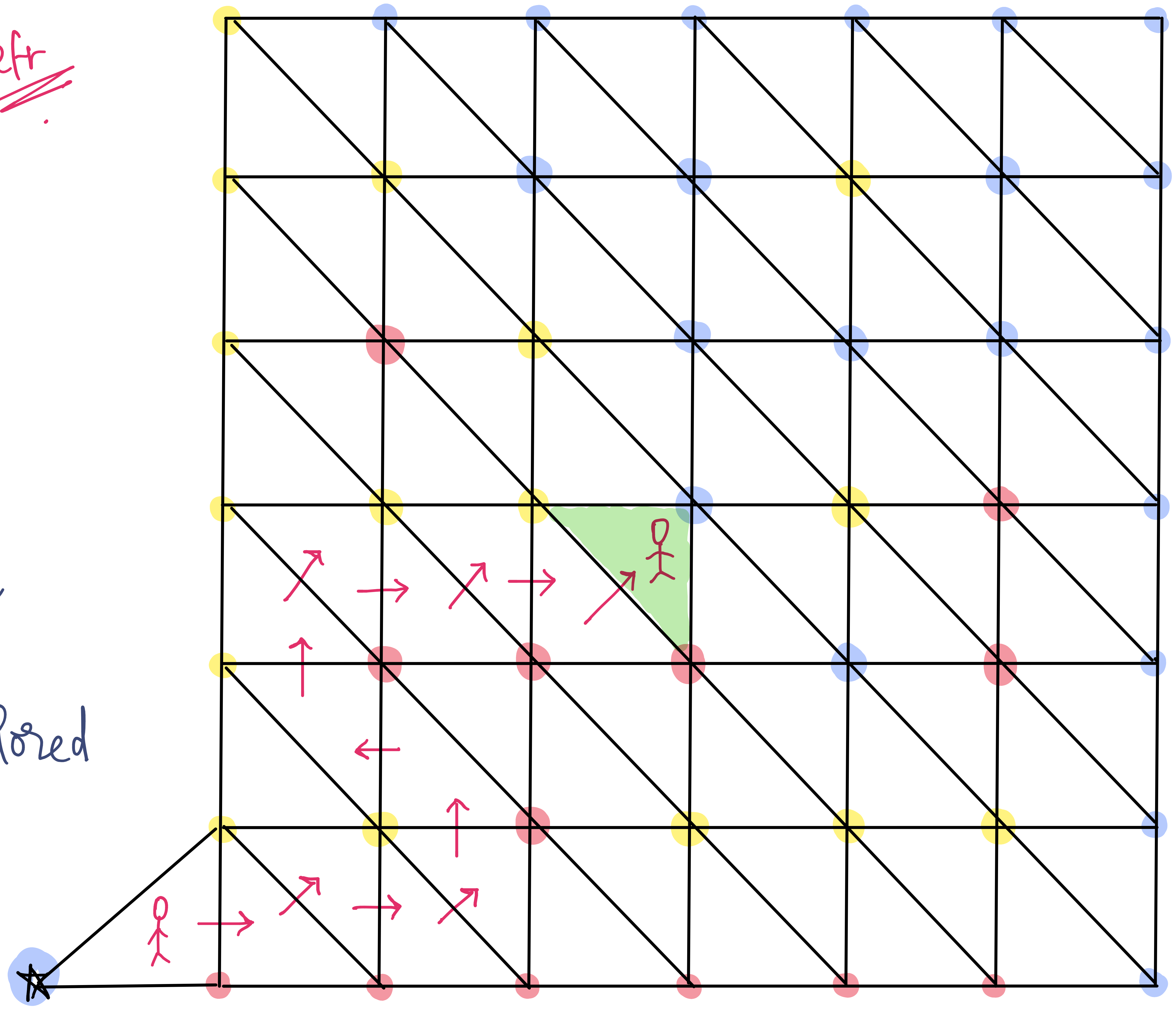

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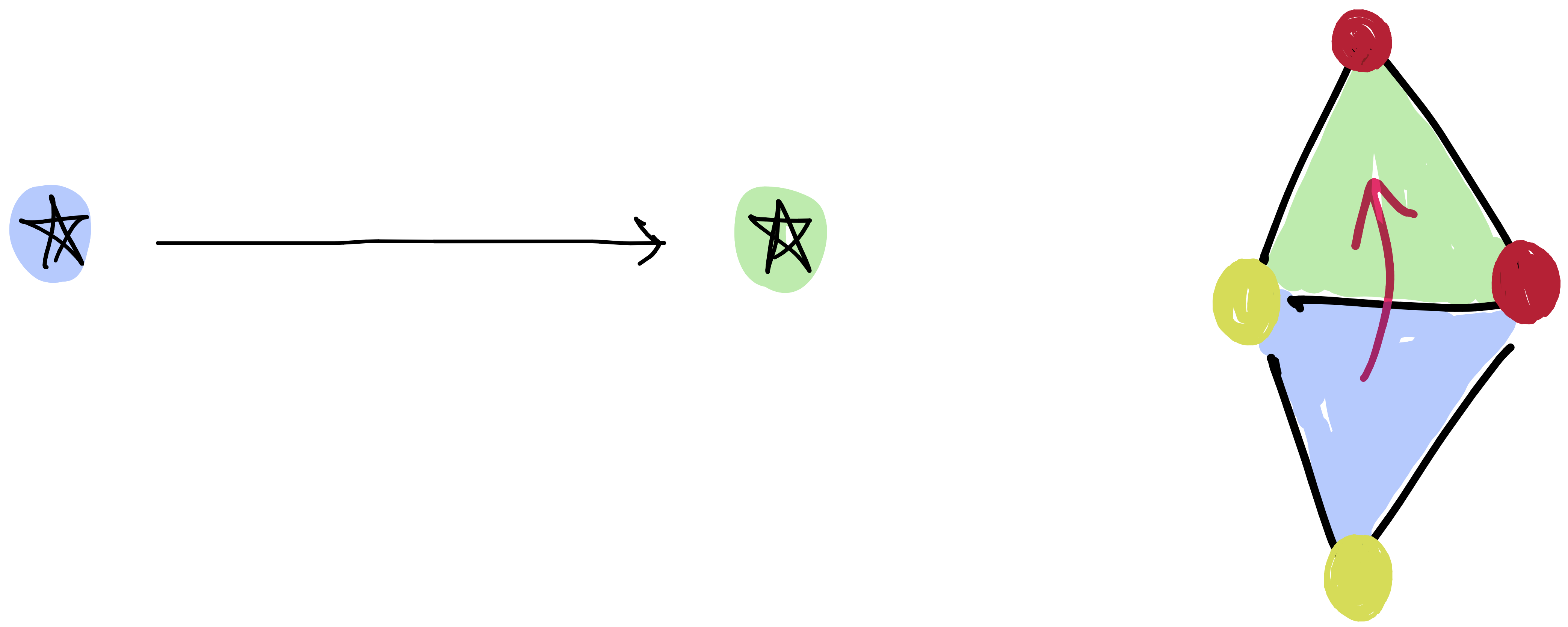
Graph G

V(G) $\rightsquigarrow$ Δ^{les} from the grid

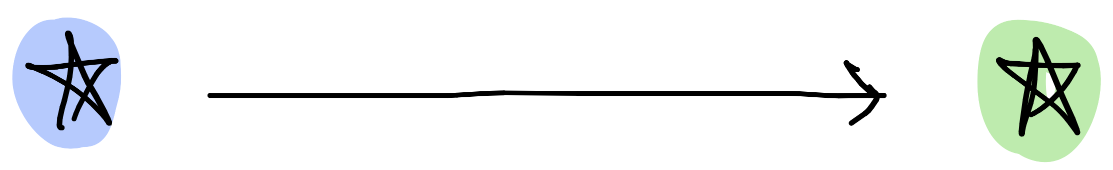


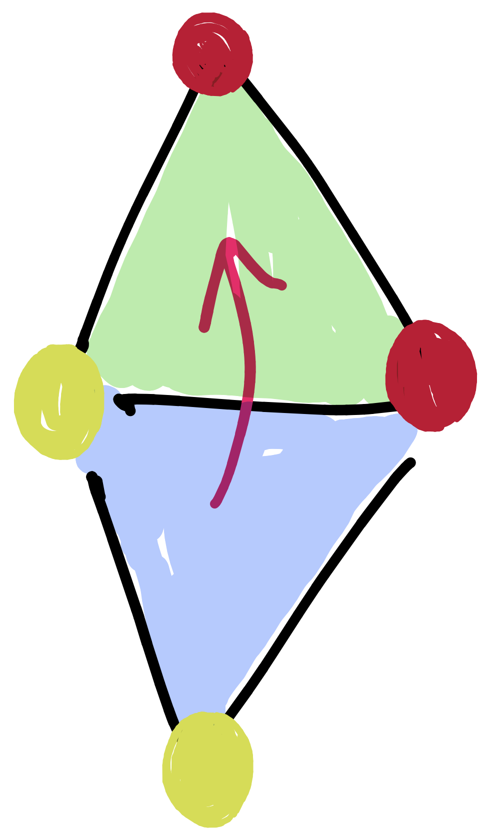
Two Δ^{le}s are connected if they share a RY door that can be crossed w/ yellow on your left.

Graph G

$V(G) \rightsquigarrow \Delta^{les}$ from the grid

$\text{indeg}(v) \leq 1$ & $\text{outdeg}(v) \leq 1$





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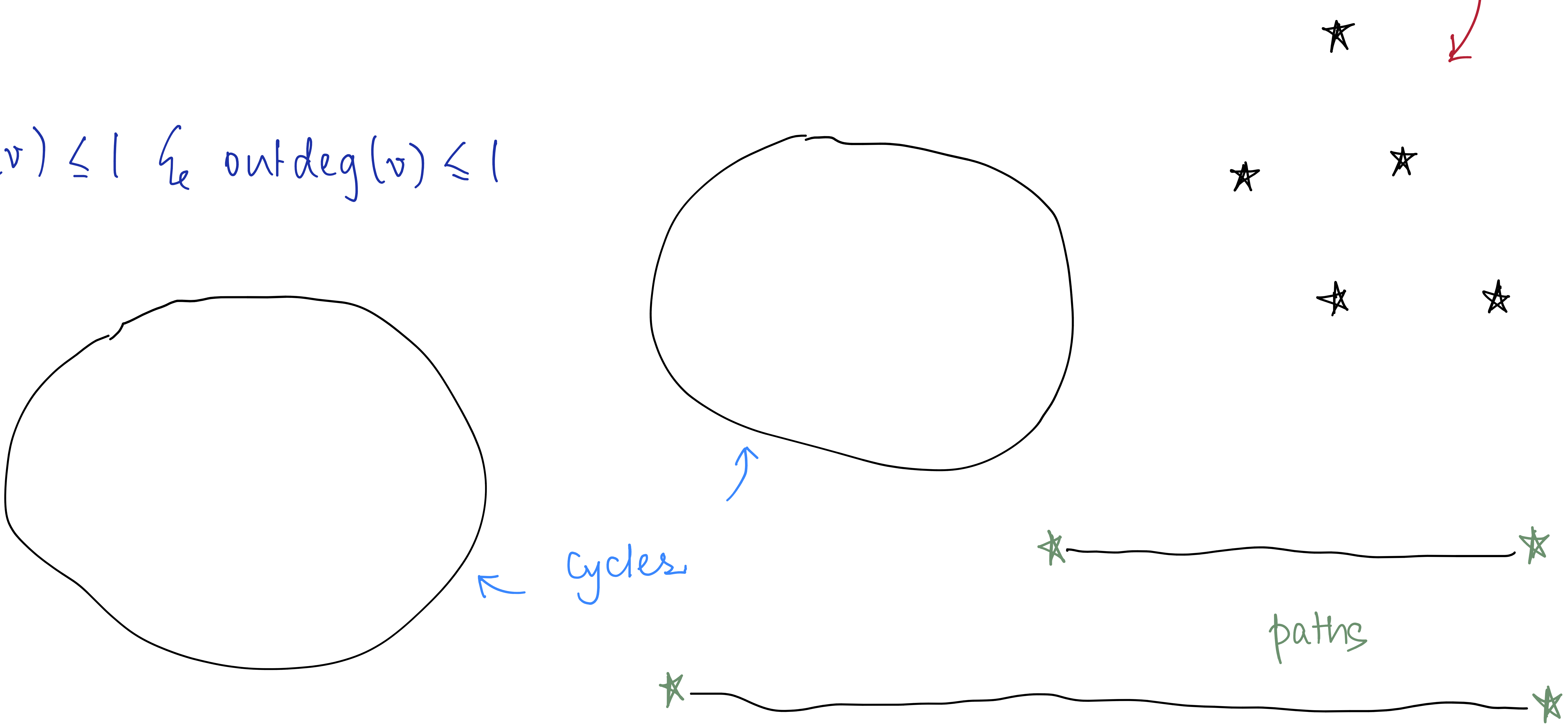
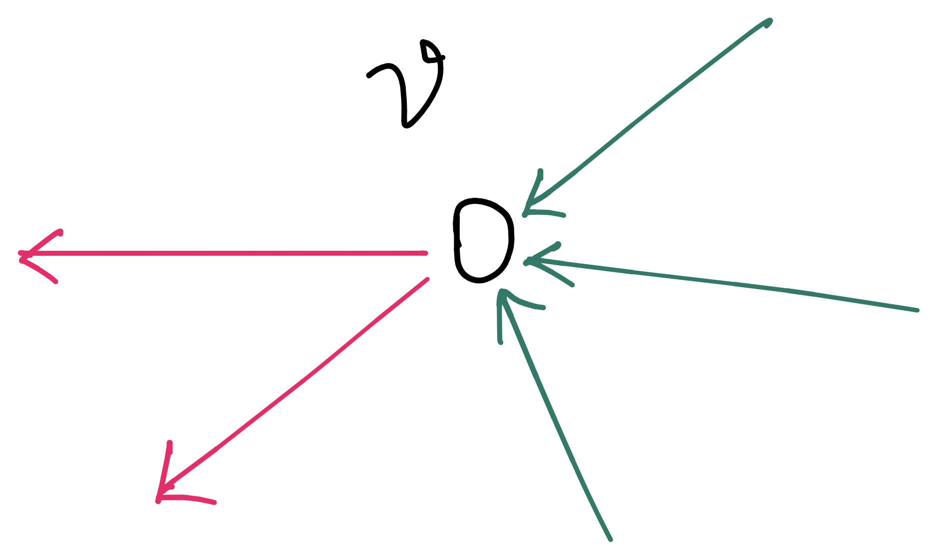
Graph G

$V(G) \rightsquigarrow \triangle$les from the grid

$\text{indeg}(v) \leq 1$ & $\text{outdeg}(v) \leq 1$



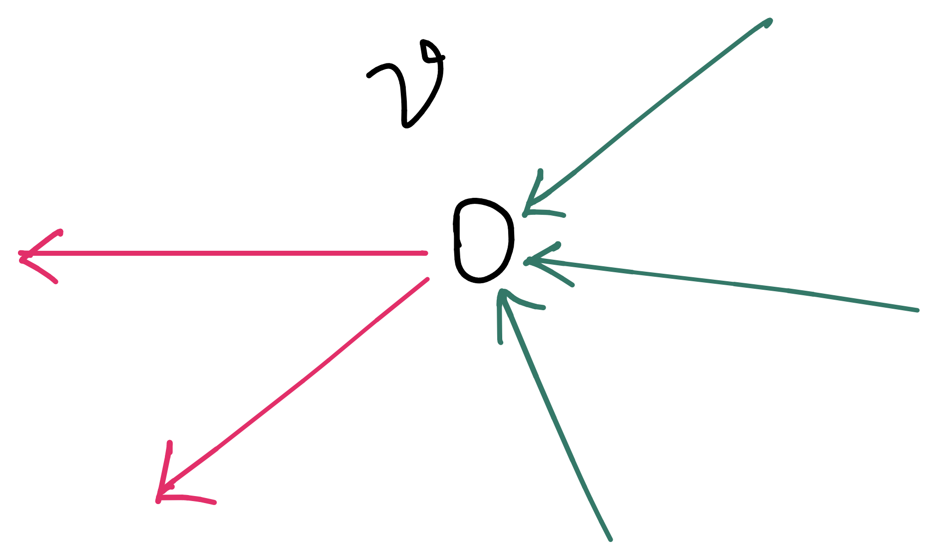


if $\exists v \in G$ s.t

v is unbalanced

(i.e, $\text{indeg}(v) \neq \text{outdeg}(v)$)

then $\exists\, u \neq v$ s.t. u is unbalanced also.



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$$\sum_{\substack{w \in V(G) \\ w \neq v}} \text{indeg}(w) + \text{indeg}(v) = |E(G)| = \sum_{\substack{w \in V(G) \\ w \neq v}} \text{outdeg}(w) + \text{outdeg}(v)$$

The PPAD Class

A graph w/ vertex set $\{0,1\}^n$ is defined by two circuits

$$\text{node id} \longrightarrow \text{PREV} \longrightarrow \text{node id}$$

$$\text{node id} \longrightarrow \text{NEXT} \longrightarrow \text{node id}$$

$$\underset{v_1}{\star} \longrightarrow \underset{v_2}{\star}$$

$$\text{PREV}(v_2) = v_1$$

$$\& \quad \text{NEXT}(v_1) = v_2$$

The PPAD Class

END-OF-THE-LINE. Given PREV & NEXT,

if 0^n is an unbalanced node,

find another unbalanced node.

PPAD $\rightsquigarrow$ $\{ L \mid L \in FNP$ & L is poly-time reducible

to the END-OF-THE-LINE problem. $\}$

Easy to check : is 0^n unbalanced ?

$\mathrm{PREV}(0^n) = x$

$\mathrm{NEXT}(0^n) = y$

if $(\mathrm{PREV}(y) = 0^n \ \& \ \mathrm{NEXT}(x) = 0^n)$ or

$(\mathrm{PREV}(y) \neq 0^n \ \& \ \mathrm{NEXT}(x) \neq 0^n)$:

0^n is balanced

o/w:

0^n is not balanced

FNP

↑

PPAD

↗

Sperner

↑

Brouwer

PPAD contains problems that we care about

$$
\begin{tikzcd}
 & \text{FNP} \\
 & \text{PPAD} \arrow[u] \\
\text{Sperner} \arrow[ur] & \\
\text{Brouwer} \arrow[u] &
\end{tikzcd}
$$

Are these problems also complete for PPAD?

PPAD contains problems that we care about

FNP

↑

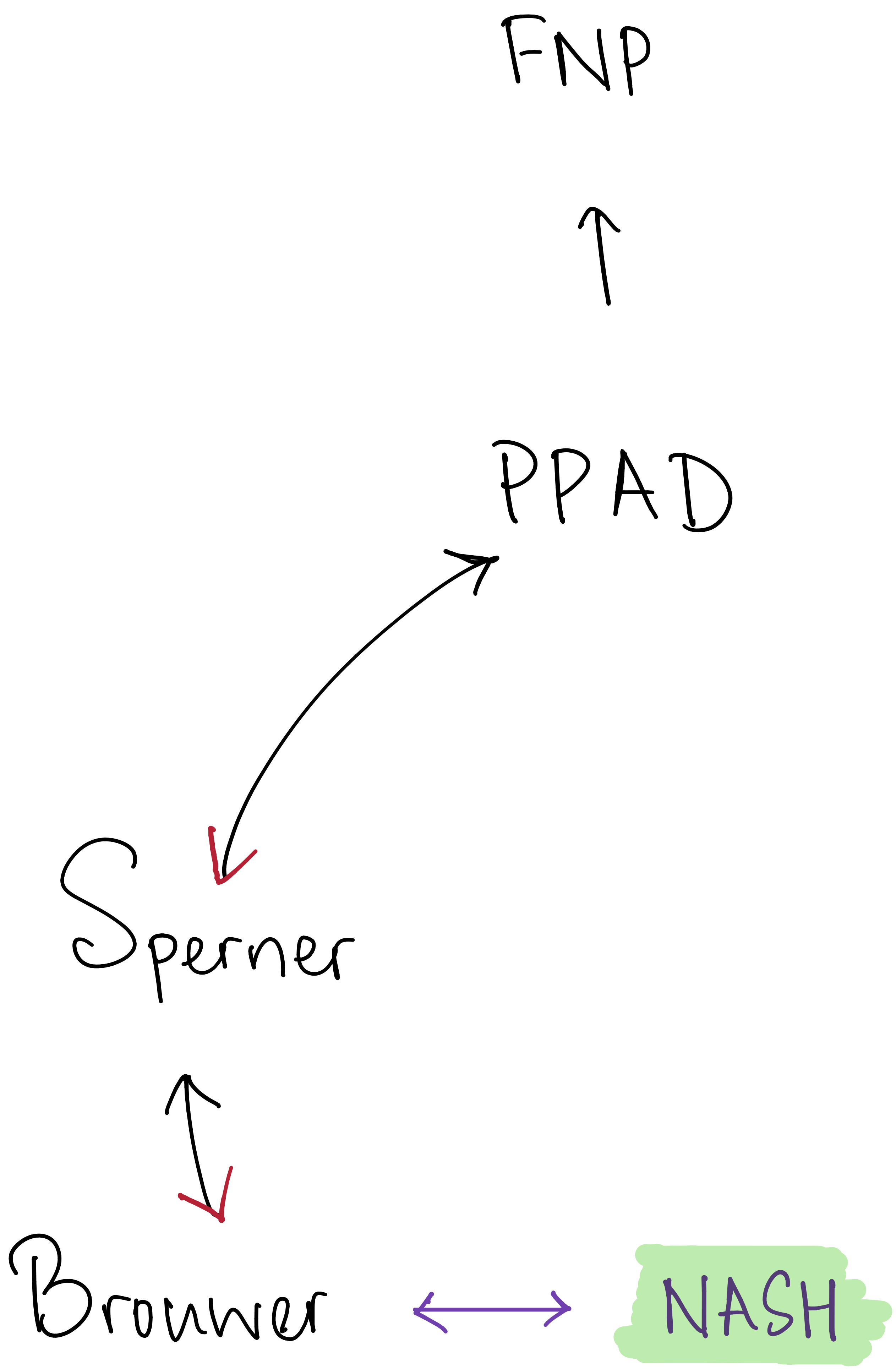
PPAD

Sperner

Brouwer

Are these problems also complete for PPAD?

PPAD contains problems that we care about



Are these problems also complete for PPAD?

PPAD contains problems that we care about

A Combinatorial Problem Which Is Complete in Polynomial Space

S. EVEN

Technion, Haifa, Israel

AND

R. E. TARJAN

Stanford University, Stanford, California

ABSTRACT This paper considers a generalization, called the Shannon switching game on vertices, of a familiar board game called Hex It is shown that determining who wins such a game if each player plays perfectly is very hard, in fact, if this game problem is solvable in polynomial time, then any problem solvable in polynomial space is solvable in polynomial time This result suggests that the theory of combinatorial games is difficult

Hex is PSPACE-complete

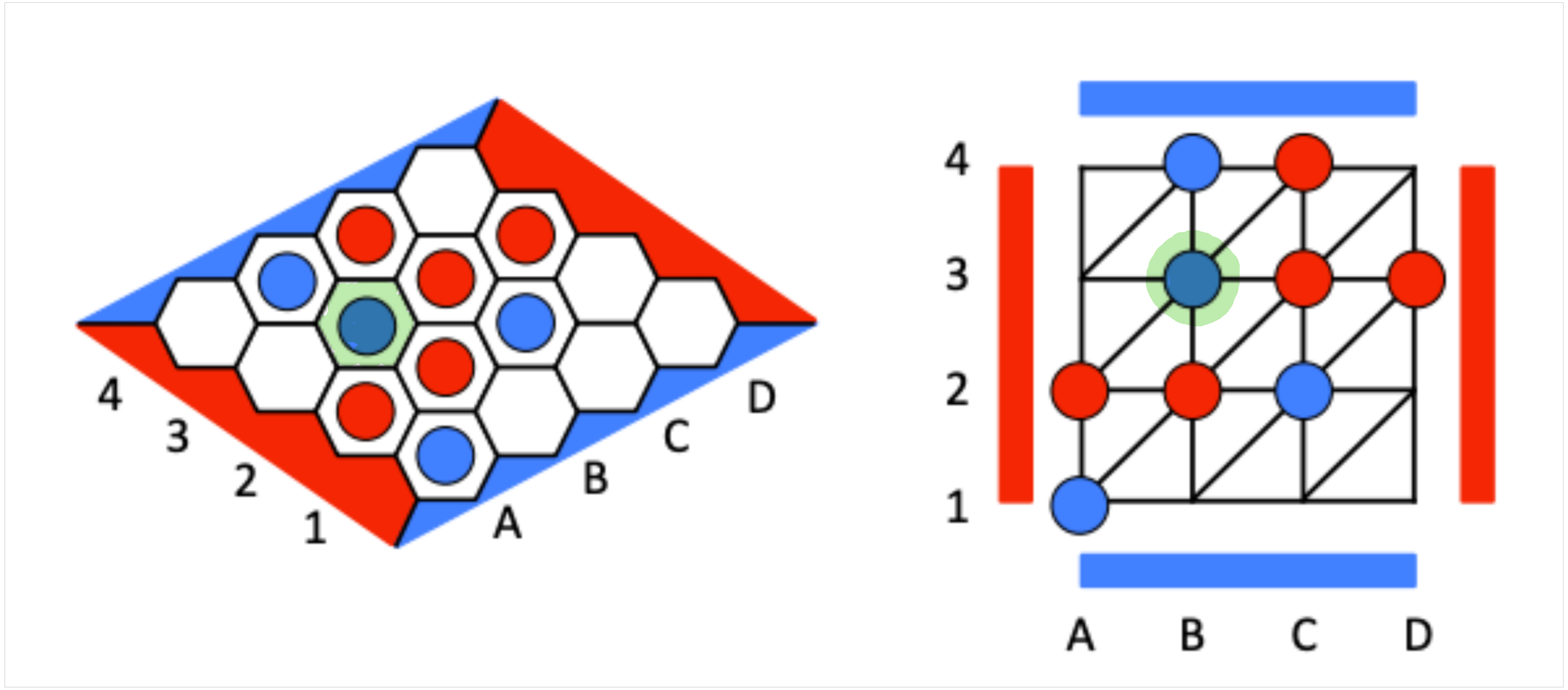
Stefan Reisch
Universität Bielefeld, Fakultät für Mathematik, Bielefeld, Germany

Hex is PSPACE-complete

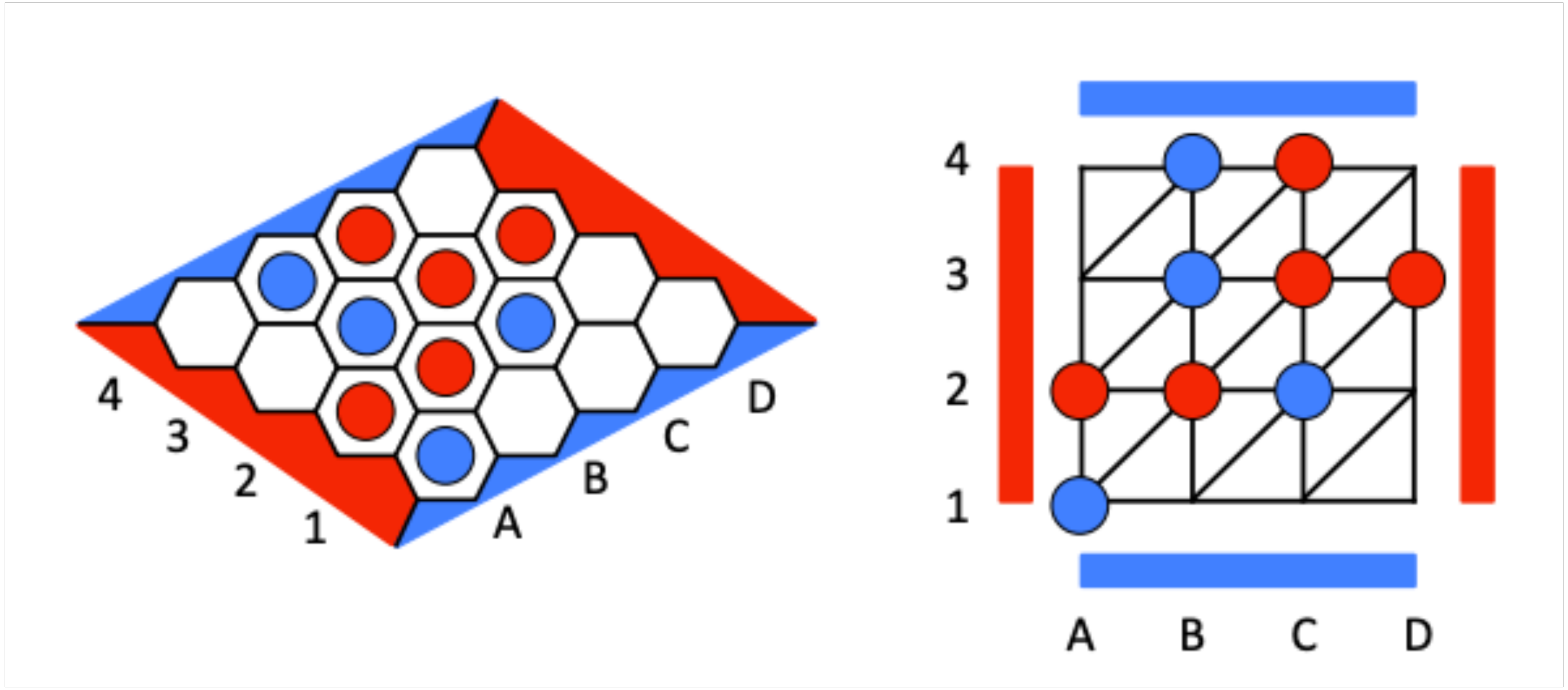
Summary. There are a number of board games such as Checkers [2], Go [5], and Gobang [8], which are known to be PSPACE-hard. This means that the problem to determine the player having a winning strategy in a given situation on an $n \times n$ board of one of these games is as hard to solve as any problem computable in polynomial space. PSPACE-completeness has been previously proven for some combinatorial games played on graphs or by logical formulas [1, 9].

In this paper we will show that the same holds for the game of Hex.

Hex & its dual graph representation



Hex & its dual graph representation

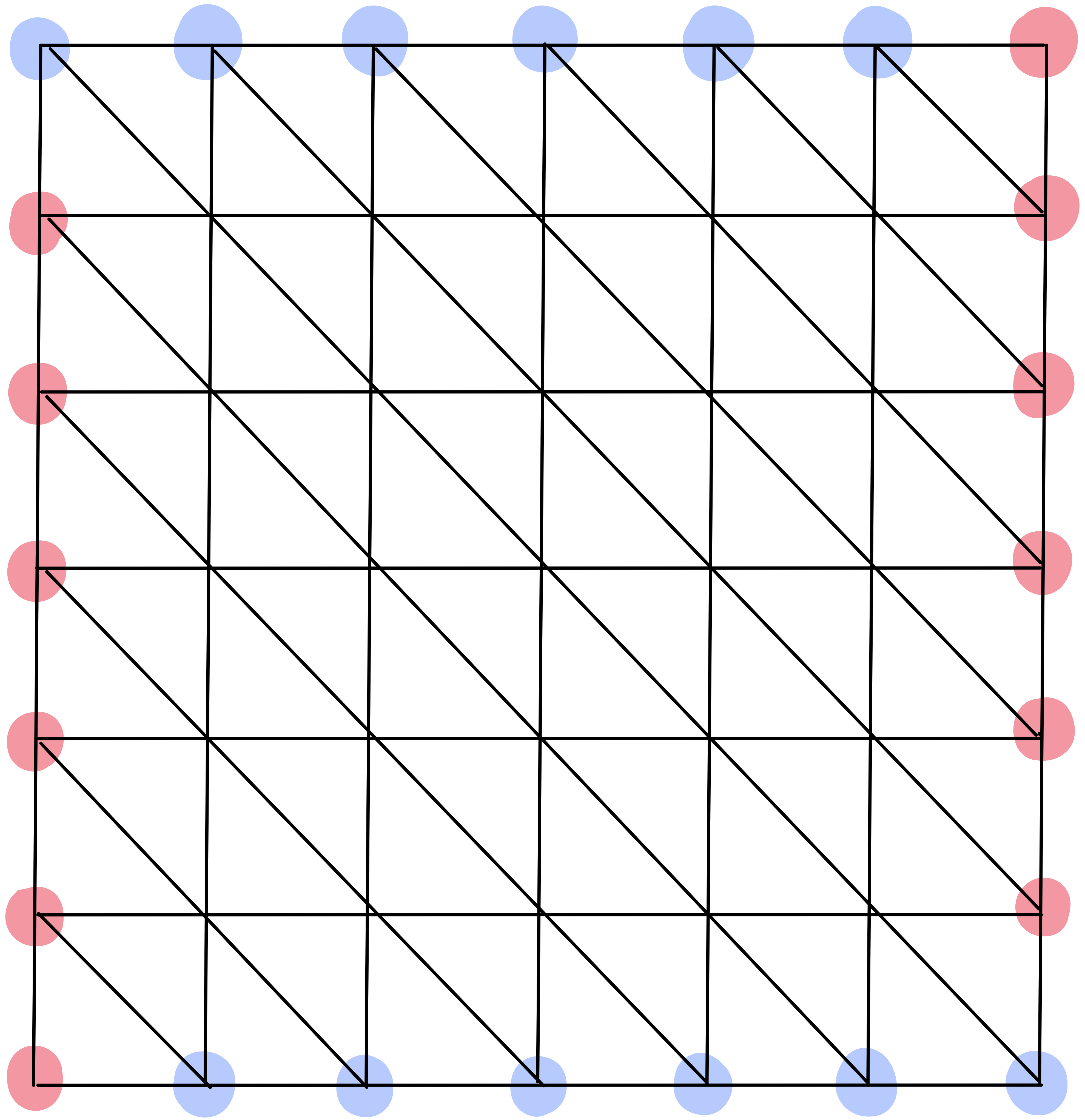


Goal.

Enter the board

w/ red on your left

& blue on your right



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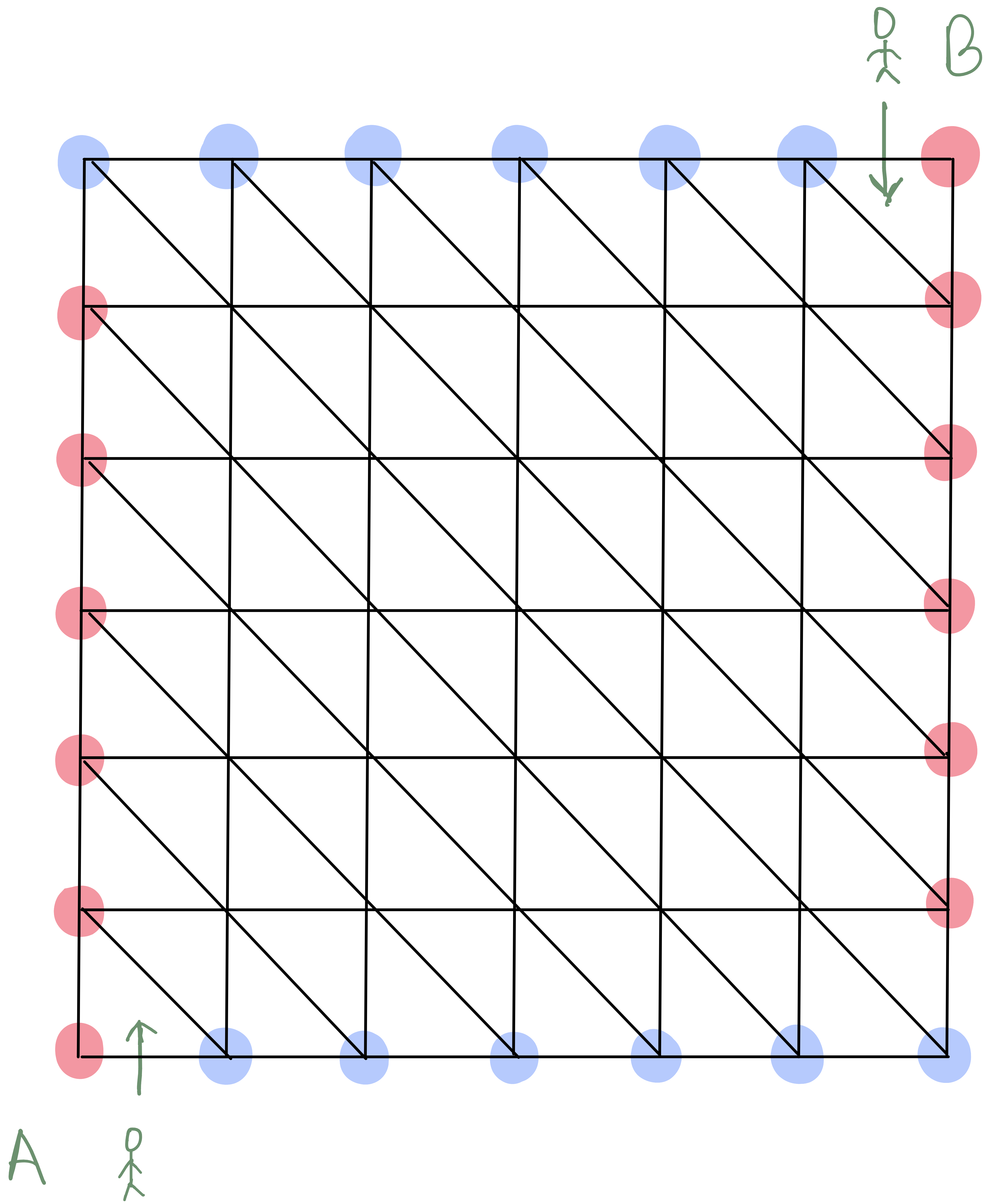

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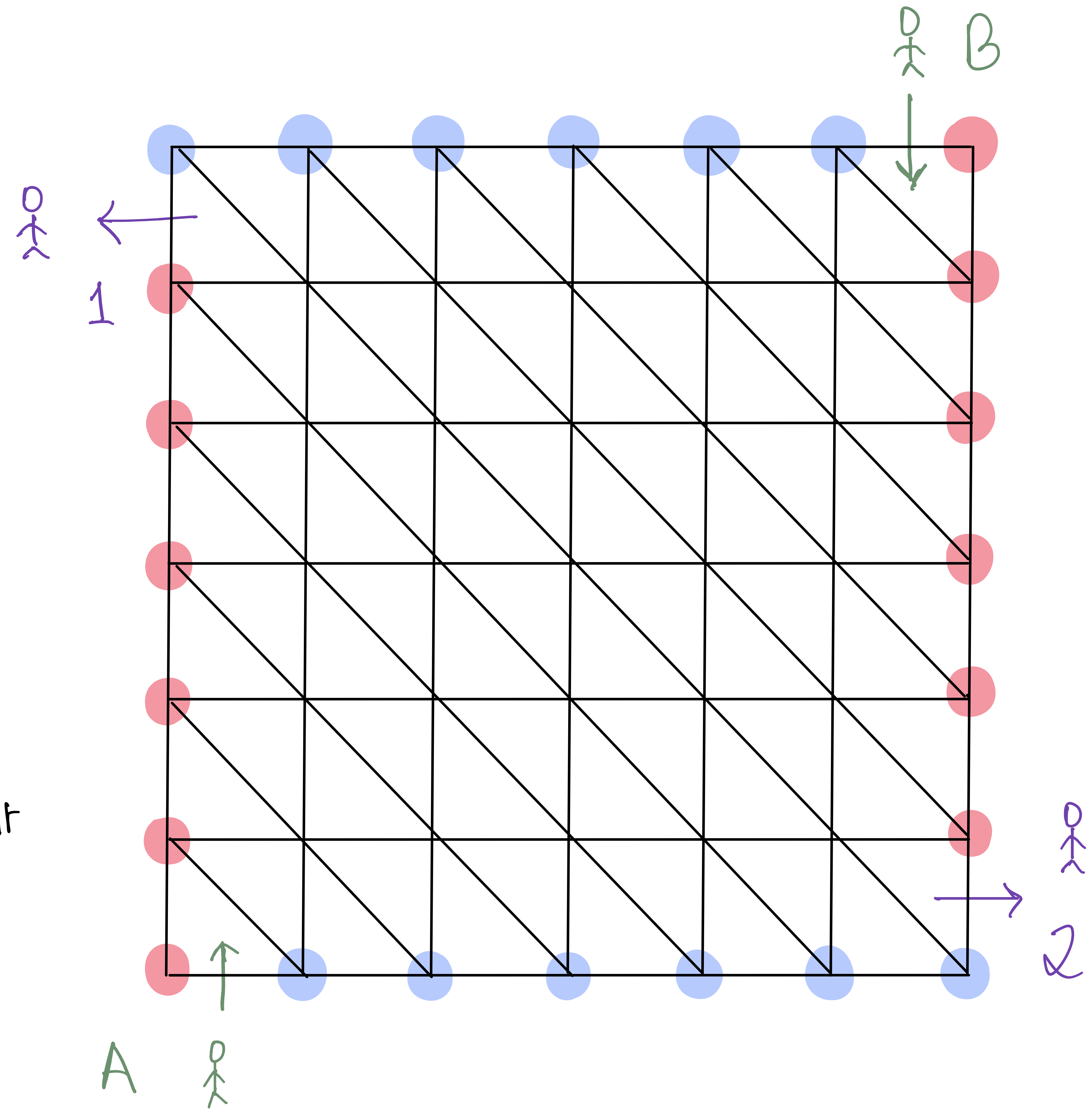

exit @ 1

→ victory for blue

exit @ 2

→ victory for red