Exam 02 AY2023-24 Semester 1

Discrete Mathematics

2023-10-12

Part 1. Multiple Choice and Short Answer Questions

Problem 1. (1 point) A house is being rewired. The house has 10 rooms named from `A` to `J`. To avoid wires getting entangled and creating short circuits, the electricians have been asked to observe the following rules.

Room `A` must be rewired before rooms `D` and `E`, Room `B` must be rewired before rooms `D` and `E`, Room `C` must be rewired before room `H`, Room `D` must be rewired before rooms `C` and `F`, Room `E` must be rewired before rooms `F` and `G`, Room `F` must be rewired before rooms `H` and `J`, Room `G` must be rewired before room `I`, Room `H` must be rewired before room `J`, Room `I` must be rewired before room `J`.

It takes one full day to rewire a room. There are enough electricians to rewire as many rooms as can be rewired in parallel, keeping in mind the constraints above. What is the minimum number of days required to complete the job?

Hint: Make a graph where the vertices correspond to rooms and dependencies correspond to directed edges. Observe that the rooms that can be rewired on the first day are the ones that have no dependencies, and take it from there.

Solution

Correct answer: 5

In general, draw up the depedency graph and iteratively delete source vertices (i.e, vertices that have no dependencies): the number of iterations will be the answer.
In this example, the following wiring scheme will work:

- Day 1: Rooms `A` and `B`
- Day 2: Rooms `D` and `E`
- Day 3: Rooms `C`, `F` and `G`
- Day 4: Rooms `H` and `I`
- Day 5: Room `J`

Problem 2. (1 point) Let G be a simple, undirected graph in which any two odd cycles intersect at a vertex. We want to claim that G can always be colored using at most k colors. What is the best (i.e, smallest) bound for k that you can come up with?

Hint: You might want to use the fact that an undirected graph is bipartite if and only if it has no odd cycles.

☐ 2 ☐ 3 ☐ 4 ■ **5** ☐ 6

Solution

Partition the vertex set of the graph into two parts: a shortest odd cycle and the rest. Notice that the rest must be biparitite because of the premise of the problem. The odd cycle can be colored with three colors (since there are no edges within the cycle — note that this is the case because we chose a shortest cycle) and the rest with two. This is the best we can hope for (consider a K_5).

Problem 3. (1 point) A *tree* is an undirected graph that is both acyclic and connected. Let T be a tree whose vertex set is properly colored with the colors black and white, and suppose there are more black vertices than white vertices. Which of the following is true?

- ☐ There is at least one white leaf.
- ■ **There is at least one black leaf.**
- ☐ There is at least one non-leaf vertex that is black.
- ☐ There are at least three black leaves.

Solution

The first, third, and last options can be ruled out by examples (consider a path on three vertices with the endpoints colored black and the middle vertex colored white). The second option can be shown to be true using a proof by contradiction: if all leaves are white, start from a white leaf and grow the tree out — we will end up having at least one distinct white vertex corresponding to every black vertex, refuting the premise of the question.

Problem 4. (1 point) The edges of K_{11} (the complete graph on eleven vertices) are colored black and white. Which of the following is true?

- ■ **Either the black or the white graph is not planar.**
- ☐ Either the black or the white graph is planar.
- ☐ Neither the black nor the white graph can be planar.
- ☐ It is possible for both the black and white subgraphs to be planar.

Solution

Use the pigeon hole principle to see that at least one of the black or white subgraphs must have at least 28 edges. Suppose WLOG that the black subgraph has 28 edges and 11 vertices. Recall that a planar graph has at most $3n - 6 = 33 - 6 = 27$ edges, so the black subgraph cannot be planar.

Please turn the page

Problem 5. (1 point) How many distinct minimum cuts does an undirected cycle on n vertices have?

Solution

$\binom{n}{2}$

Any pair of edges corresponds to a minimum cut, these are the only minimum cuts, and there are n edges.

Problem 6. (1 point) You flip a fair coin and toss a fair six-sided die. What is the probability that the coin lands on heads, the die lands on 4, **or** both events occur?

Solution

Correct answer: 7/12

The probability that the coin lands on heads is $\frac{1}{2}$. The probability that the die lands on 4 is $\frac{1}{6}$. The probability that the coin lands on heads and the die lands on 4 is $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.
Hence, by the inclusion-exclusion principle, the probability that the coin lands on heads or the die lands on 4 is

$$\frac{1}{2} + \frac{1}{6} - \frac{1}{12} = \frac{7}{12}.$$

This is counting heads, 4 , and both, even though the probability of both is being subtracted.

Problem 7. (1 point) A high school offers courses in Sanskrit and Gujarati (and only those two languages), and students can enroll in both if they like. A sample of students from the school were asked which language courses they enrolled in:

- 60% of the students responded that they took Sanskrit.
- 30% of the students responded that they took Gujarati.
- 20% of the students responded that they did not enroll in any language course.

A student is randomly chosen from among the students surveyed.

What is the probability that the student enrolled in the Sanskrit course but not the Gujarati course?

Solution

Correct answer: `1/2` or `0.5`

If 20% of students are not in a language course, then 80% of the students are in a language course. Now, 60% are in Sanskrit and 30% are in Gujarati, so that means that 10% must be enrolled in both. Then 50% of students are enrolled in the Sanskrit course but not the Gujarati course. Thus, the probability that a student would be enrolled in the Sanskrit course but not the Gujarati course is 0.5.

Problem 8. Total: 3

A flu has hit a fictional town called Derry with a population of 10,000. Already, 500 people have fallen ill, while others haven't been affected yet. Our imaginary friend Kyzeel lives in Derry. Notice that Kyzeel has only a 5% chance of being ill.

(8.a) (1 point) It turns out that Kyzeel's home test comes back positive! According to the box, the test is correct 90% of the time. Let's figure out the odds of Kyzeel being healthy even with a test result saying he has the flu. The test is 90% accurate, meaning 90% of 500 sick townsfolk will correctly test positive.

How many townsfolk actually have the flu and also test positive for it?

Solution

450

500 townsfolk have the flu. 90% of these will take the test and get an accurate result saying they have the flu. That is,

$$\frac{90}{100} \cdot 500 = 450$$

sick townsfolk will get correct positive test results.

(8.b) (1 point) Now, Kyzeel could be one of the sick townsfolk that test positive, but he could also be among the healthy townsfolk who get a wrong test result. How many healthy townsfolk should expect to get test results saying they have the flu?

Solution

950

9,500 townsfolk do not have the flu. 90% of these will take the test and get the accurate result saying they do not have the flu. However, the remaining 10% of these healthy townsfolk will get a wrong test result. That is,

$$\frac{10}{100} \cdot 9,500 = 950$$

healthy townsfolk will get test results saying they are sick when they actually are not.

(8.c) (1 point) We know for sure that Kyzeel is one of the townsfolk in Derry who test positive for the flu. We want the chance that Kyzeel is healthy conditional on his positive test result:

$$\mathbb{P}(\odot \mid +),$$

where the icon to the left denotes the event we're interested in (Kyzeel is healthy), while the one on the right is the condition (positive test result +). What is the value of this probability?

Solution

68%

We can expect $450 + 950 = 1,400$ townsfolk to get a positive test result, with Kyzeel being one of them. We can also expect 950 of those townsfolk to actually be healthy. The probability that Kyzeel is healthy while getting a positive result is

$$\frac{950}{1,400} \approx 68\%$$

Part 2. Subjective Questions

Problem 9. Total: 5

Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $[n]$ and each set is of size l. We now want to color each element of $[n]$ with either red or blue, and ensure that there is no set A_i such that all its elements get the same color. We will now prove that if $m < 2^{l-1}$, then there exists a coloring such that for every set A_i has two elements with different colors. We will prove this in two steps using the probabilistic method:

(9.a) (2 points) Every element of $[n]$ is colored uniformly at random (probability of blue is $1/2$) and is independent of the colors given to the other elements. Let E_i be the event that all elements in A_i get the same color. Prove that

$$p(E_i) = 1/2^{l-1}.$$

Solution

We have

$$\begin{aligned} p(E_i) &= p(\text{ all blue }) + p(\text{ all red }) \\ &= \frac{1}{2^l} + \frac{1}{2^l} \\ &= \frac{1}{2^{l-1}}. \end{aligned}$$

(9.b) (3 points) Using the union bound, show that if $m < 2^{l-1}$, then the probability that there is a set A_i whose elements get the same color is less than 1. Also, complete the proof of the theorem we set out to prove.

Solution

We want to show that $p(E_1 \cup E_2 \cup \ldots \cup E_n) < 1$. By the union bound

$$p(E_1 \cup E_2 \cup \ldots \cup E_n) \leq \sum_{i=1}^{m} p(E_i) \leq \frac{m}{2^{l-1}} < 1,$$

where we used the fact that $m < 2^{l-1}$. We are now ready to finish the proof of the theorem. Since $p(E_1 \cup E_2 \cup \ldots \cup E_n) < 1$, we know that there exists a coloring such that every E_i does not happen. This translates to saying that there exists a coloring in which every set A_i has two elements with different colors.

Problem 10. (5 points) Recall that a matching in a graph $G = (V, E)$ is a collection of vertex-disjoint edges, i.e, it is a subset $F \subseteq E$ such that any pair of edges in F do *not* share an endpoint. Hall's theorem states the following:

Let G be an undirected bipartite graph with bipartition (V_1, V_2). The graph G has a matching saturating V_1 if and only if for all $X \subseteq V_1$, we have $|N(X)| \geq |X|$.

Note that $|V_1| \leqslant |V_2|$, and a matching saturating V_1 is simply a matching which is such that every vertex of V_1 is an endpoint of one of the edges in the matching.

Now, a magician and her assistant are performing the following magic trick. A volunteer from the audience picks five cards from a standard deck of 52 cards and then passes the deck to the assistant. The assistant shows to the magician, one by one in some order, four cards from the chosen set of five cards. Then, the magician guesses the remaining fifth card.

Show, *using Hall's theorem*, that this magic trick can be performed without any help of magic.

Note that this trick has been performed in a tutorial, and an explicit strategy has also been discussed. This question is about modeling the trick as a biparite graph in such a way that a matching corresponds to a strategy.

Solution

Consider the following bipartite graph: on one side (say L) there are all $\begin{pmatrix} 52 \\ 5 \end{pmatrix}$ sets of five cards (possibly chosen by the volunteer), and on the other side (say R) there are all $52 \cdot 51 \cdot 50 \cdot 49$ tuples of pairwise different four cards (possibly shown by the assistant). A set S is adjacent to a tuple T if all cards of T belong to S. Using Hall's theorem, we now show that this graph admits a matching saturating the side with all sets of five cards. This matching induces a strategy for the assistant and the magician.
Observe that:

- every vertex in L has $\binom{5}{4} \cdot 4! = 120$ neighbors in R
- every vertex in R has $52 - 4 = 48$ neighbors in L.

Consider any subset S of L. By double counting the edges between $S \subseteq L$ and $N(S) \subseteq R$, we have:

$$120|S| = |E(S, N(S))| \leqslant 48|N(S)| \Rightarrow |N(S)| \geqslant |S|,$$

since every vertex in S is incident to 120 edges, and if $N(S)$ has t vertices, then it has $48t$ edges incident on it, of which the edges coming from S are a subset (hence the second part of the relationship above is an inequality).
The explicit strategy used in the class was the following. In every set of five cards, there are two cards of the same color, say a and b. Moreover, as there are 13 cards of the same color, the cards a and b differ by at most 6 , that is, $a + i = b$ or $b + i = a$ for some $1 \leq i \leq 6$, assuming some cyclic order on the cards of the same color. Without loss of generality, assume $a + i = b$. The assistant first shows the card a to the magician. Then, using the remaining three cards, and some fixed total order on the whole deck of cards, the assistant shows the integer i (there are $3! = 6$ permutations of remaining three cards). Consequently, the magician knows the card b by knowing its color (the same as the first card show by the assistant) and the value of the card a and the number i.

Question	Points	Score
Problem 1	1	
Problem 2	1	
Problem 3	1	
Problem 4	1	
Problem 5	1	
Problem 6	1	
Problem 7	1	
Problem 8	3	
Problem 9	5	
Problem 10	5	
Total:	20	

End of the exam