ES214 Name [ ] IIT Gandhinagar

Roll Number [ ]

Exam 02 AY2023-24 Semester 1

Discrete Mathematics

2023-10-12

Part 1. Multiple Choice and Short Answer Questions

Problem 1. (1 point) A house is being rewired. The house has 10 rooms named from `A` to `J`. To avoid wires getting entangled and creating short circuits, the electricians have been asked to observe the following rules.

Room `A` must be rewired before rooms `D` and `E`, Room `B` must be rewired before rooms `D` and `E`, Room `C` must be rewired before room `H`, Room `D` must be rewired before rooms `C` and `F`, Room `E` must be rewired before rooms `F` and `G`, Room `F` must be rewired before rooms `H` and `J`, Room `G` must be rewired before room `I`, Room `H` must be rewired before room `J`, Room `I` must be rewired before room `J`.

It takes one full day to rewire a room. There are enough electricians to rewire as many rooms as can be rewired in parallel, keeping in mind the constraints above. What is the minimum number of days required to complete the job?

Hint: Make a graph where the vertices correspond to rooms and dependencies correspond to directed edges. Observe that the rooms that can be rewired on the first day are the ones that have no dependencies, and take it from there.

[ ]

Problem 2. (1 point) Let G be a simple, undirected graph in which any two odd cycles intersect at a vertex. We want to claim that G can always be colored using at most k colors. What is the best (i.e, smallest) bound for k that you can come up with?

Hint: You might want to use the fact that an undirected graph is bipartite if and only if it has no odd cycles.

☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

Problem 3. (1 point) A *tree* is an undirected graph that is both acyclic and connected. Let T be a tree whose vertex set is properly colored with the colors black and white, and suppose there are more black vertices than white vertices. Which of the following is true?

- ☐ There is at least one white leaf.
- ☐ There is at least one black leaf.
- ☐ There is at least one non-leaf vertex that is black.
- ☐ There are at least three black leaves.

Problem 4. (1 point) The edges of K_{11} (the complete graph on eleven vertices) are colored black and white. Which of the following is true?

- ☐ Either the black or the white graph is not planar.
- ☐ Either the black or the white graph is planar.
- ☐ Neither the black nor the white graph can be planar.
- ☐ It is possible for both the black and white subgraphs to be planar.

Problem 5. (1 point) How many distinct minimum cuts does an undirected cycle on n vertices have?

Problem 6. (1 point) You flip a fair coin and toss a fair six-sided die. What is the probability that the coin lands on heads, the die lands on 4, **or** both events occur?

Problem 7. (1 point) A high school offers courses in Sanskrit and Gujarati (and only those two languages), and students can enroll in both if they like. A sample of students from the school were asked which language courses they enrolled in:

- 60% of the students responded that they took Sanskrit.
- 30% of the students responded that they took Gujarati.
- 20% of the students responded that they did not enroll in any language course.

A student is randomly chosen from among the students surveyed.

What is the probability that the student enrolled in the Sanskrit course but not the Gujarati course?

Problem 8. Total: 3

A flu has hit a fictional town called Derry with a population of 10,000. Already, 500 people have fallen ill, while others haven't been affected yet. Our imaginary friend Kyzeel lives in Derry. Notice that Kyzeel has only a 5% chance of being ill.

(8.a) (1 point) It turns out that Kyzeel's home test comes back positive! According to the box, the test is correct 90% of the time. Let's figure out the odds of Kyzeel being healthy even with a test result saying he has the flu. The test is 90% accurate, meaning 90% of 500 sick townsfolk will correctly test positive. How many townsfolk actually have the flu and also test positive for it?

(8.b) (1 point) Now, Kyzeel could be one of the sick townsfolk that test positive, but he could also be among the healthy townsfolk who get a wrong test result. How many healthy townsfolk should expect to get test results saying they have the flu?

(8.c) (1 point) We know for sure that Kyzeel is one of the townsfolk in Derry who test positive for the flu. We want the chance that Kyzeel is healthy conditional on his positive test result:

$$\mathbb{P}(\odot \mid +),$$

where the icon to the left denotes the event we're interested in (Kyzeel is healthy), while the one on the right is the condition (positive test result +). What is the value of this probability?

Name

Roll Number

Part 2. Subjective Questions

Problem 9. Total: 5

Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $[n]$ and each set is of size l. We now want to color each element of $[n]$ with either red or blue, and ensure that there is no set A_i such that all its elements get the same color. We will now prove that if $m < 2^{l-1}$, then there exists a coloring such that for every set A_i has two elements with different colors. We will prove this in two steps using the probabilistic method:

(9.a) (2 points) Every element of $[n]$ is colored uniformly at random (probability of blue is $1/2$) and is independent of the colors given to the other elements. Let E_i be the event that all elements in A_i get the same color. Prove that

$$p\left(E_i\right) = 1/2^{l-1}.$$

(9.b) (3 points) Using the union bound, show that if $m < 2^{l-1}$, then the probability that there is a set A_i whose elements get the same color is less than 1. Also, complete the proof of the theorem we set out to prove.

Problem 10. (5 points) Recall that a matching in a graph $G = (V, E)$ is a collection of vertex-disjoint edges, i.e, it is a subset $F \subseteq E$ such that any pair of edges in F do *not* share an endpoint. Hall's theorem states the following:

Let G be an undirected bipartite graph with bipartition (V_1, V_2). The graph G has a matching saturating V_1 if and only if for all $X \subseteq V_1$, we have $|N(X)| \geq |X|$.

Note that $|V_1| \leqslant |V_2|$, and a matching saturating V_1 is simply a matching which is such that every vertex of V_1 is an endpoint of one of the edges in the matching.

Now, a magician and her assistant are performing the following magic trick. A volunteer from the audience picks five cards from a standard deck of 52 cards and then passes the deck to the assistant. The assistant shows to the magician, one by one in some order, four cards from the chosen set of five cards. Then, the magician guesses the remaining fifth card.

Show, *using Hall's theorem*, that this magic trick can be performed without any help of magic.

Note that this trick has been performed in a tutorial, and an explicit strategy has also been discussed. This question is about modeling the trick as a biparite graph in such a way that a matching corresponds to a strategy.