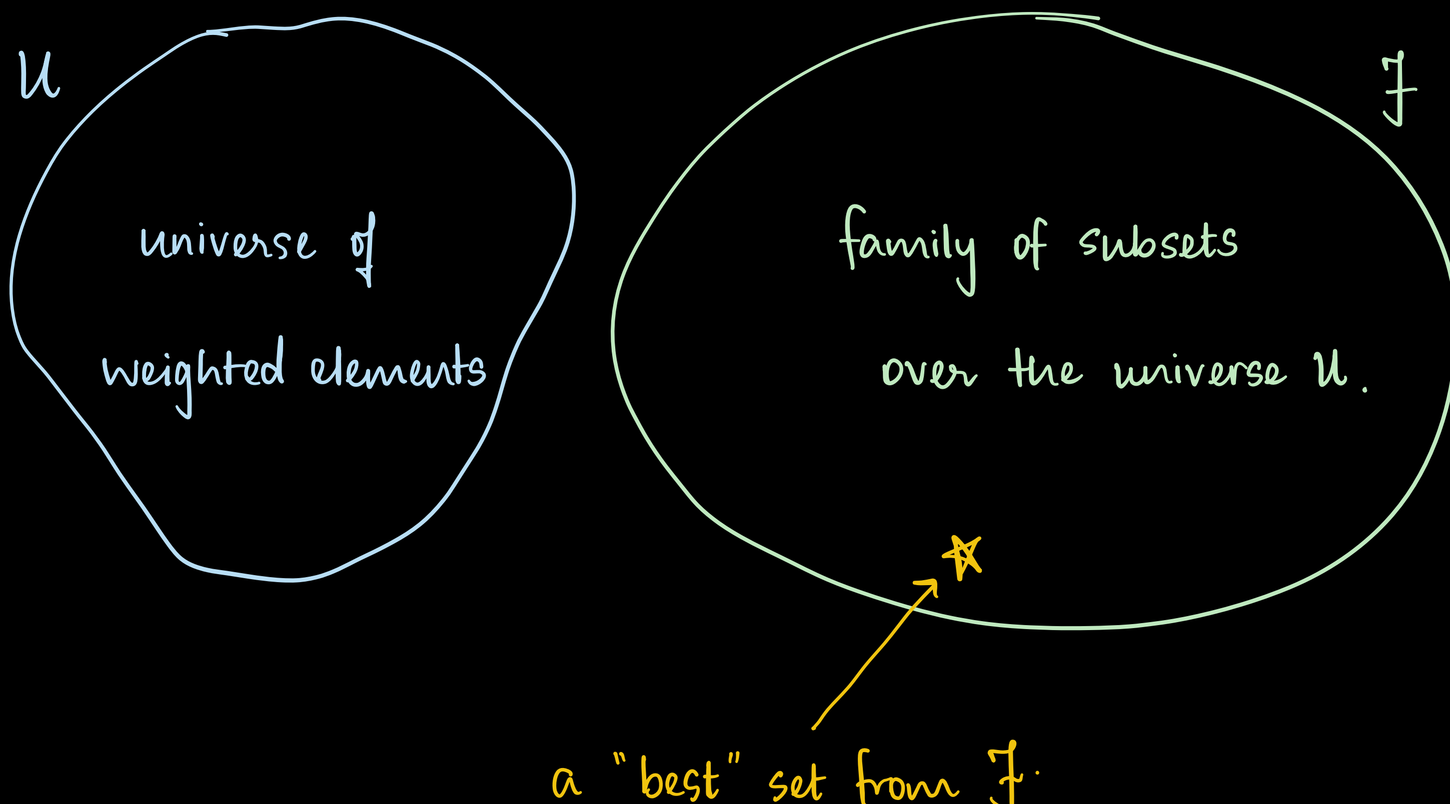
Advanced Algorithms

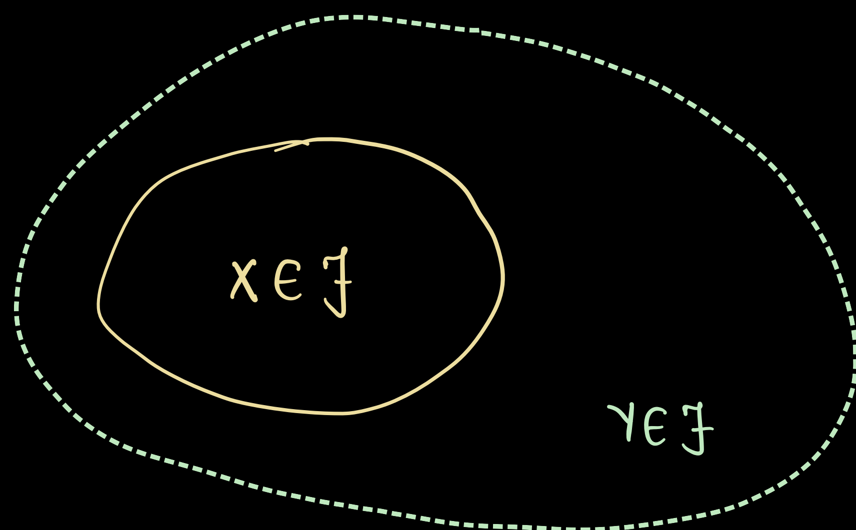
In this video:

1. Define an optimization problem
2. Attempt a greedy algorithm
3. Figure out when greedy works
4. Define the notion of a matroid

A natural optimization problem.

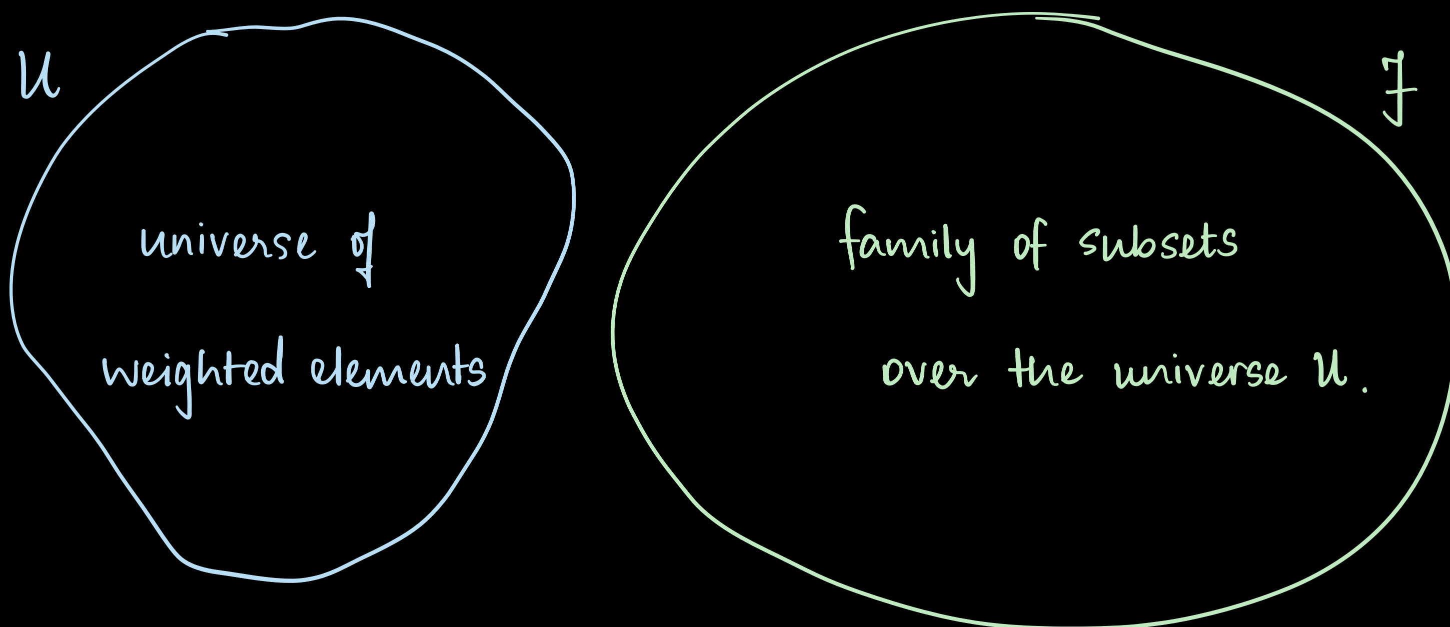


Typically interested in maximal sets



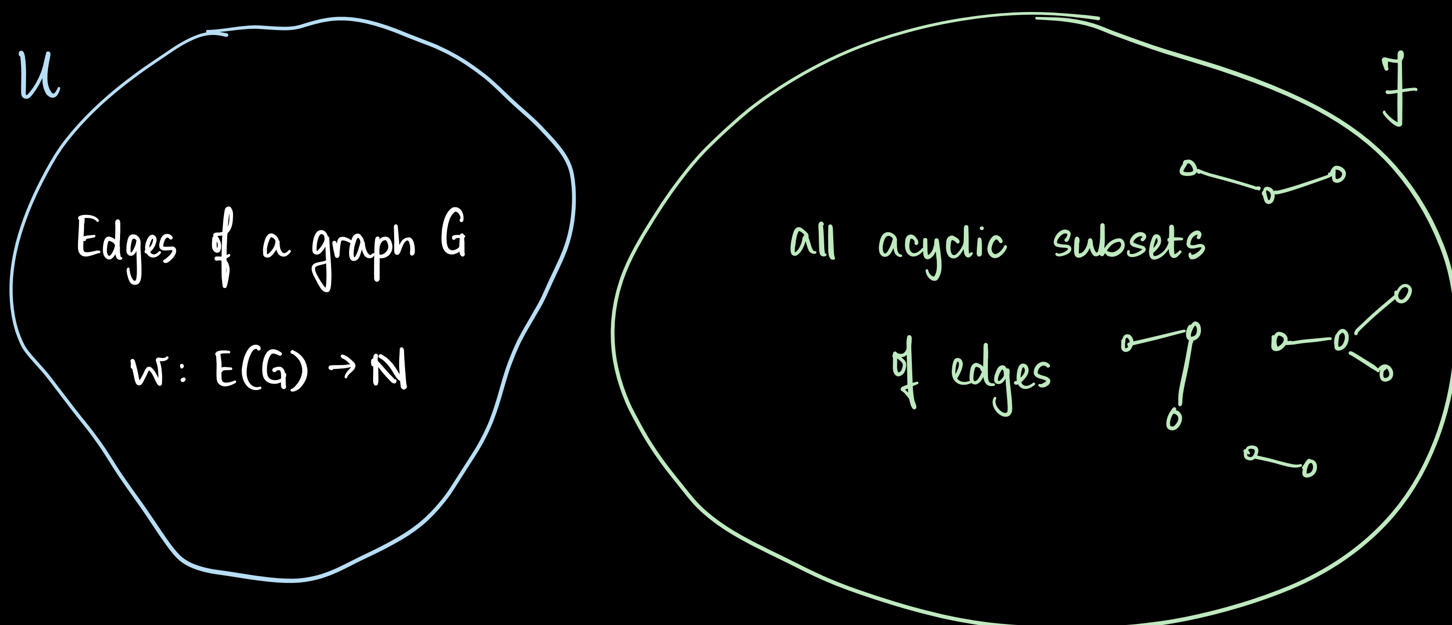
If $\exists\, Y \supseteq X$, then X is not a valid answer.

A natural optimization problem.



WANT: a maximal set from $\mathcal{F}$ with largest / smallest weight.

A natural optimization problem.



Want: a minimum spanning tree

A natural optimization problem.

$\mathcal{U}$

a set of tasks* w/ deadlines & costs

$\mathcal{F}$

subsets of tasks that can be completed on time

* n tasks
  n days
& one task per day

Want: The most expensive collection of feasible tasks.

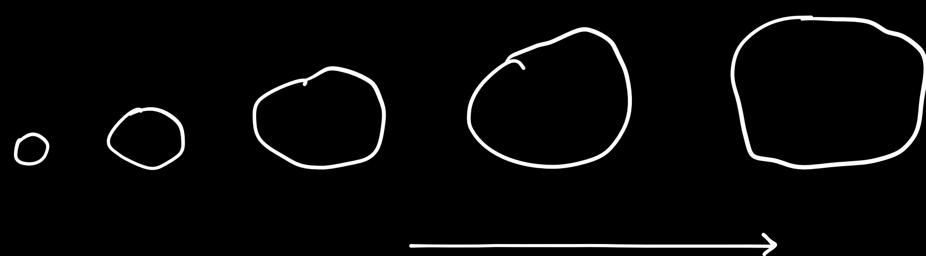
Brute-force $\rightsquigarrow$ run through all sets in $\mathcal{F}$.

exponential in $|U|$.

Greedy Approach.

$S = \emptyset$

Sort elements in increasing order of weight.



For i in 1 to n:

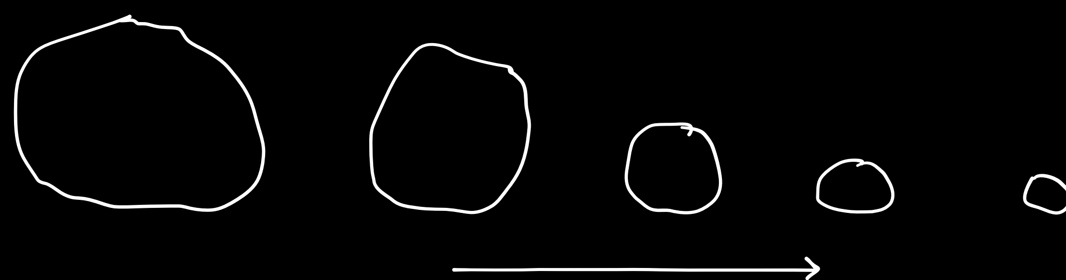
  if $S \cup \{ i^{th} \text{ element} \} \in \mathcal{I}$

    add the i^{th} element to the solution

Greedy Approach.

$S = \emptyset$

Sort elements in decreasing order of weight.



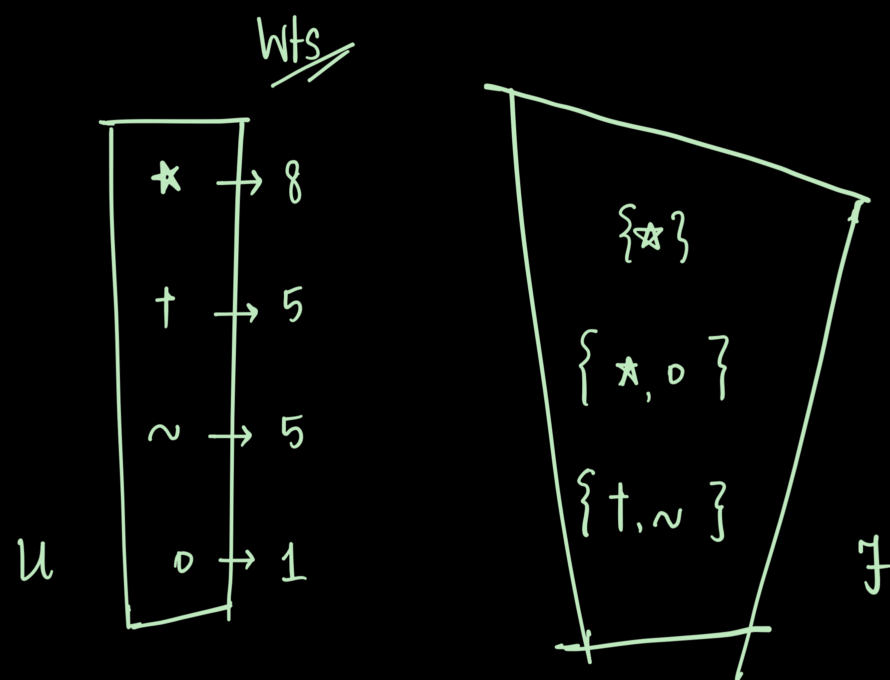
For i in 1 to n:

  if $S \cup \{i^{th} \text{ element}\} \in \mathcal{F}$

    add the i^{th} element to the solution

Does the greedy algorithm work
on all possible inputs?

Counter-example



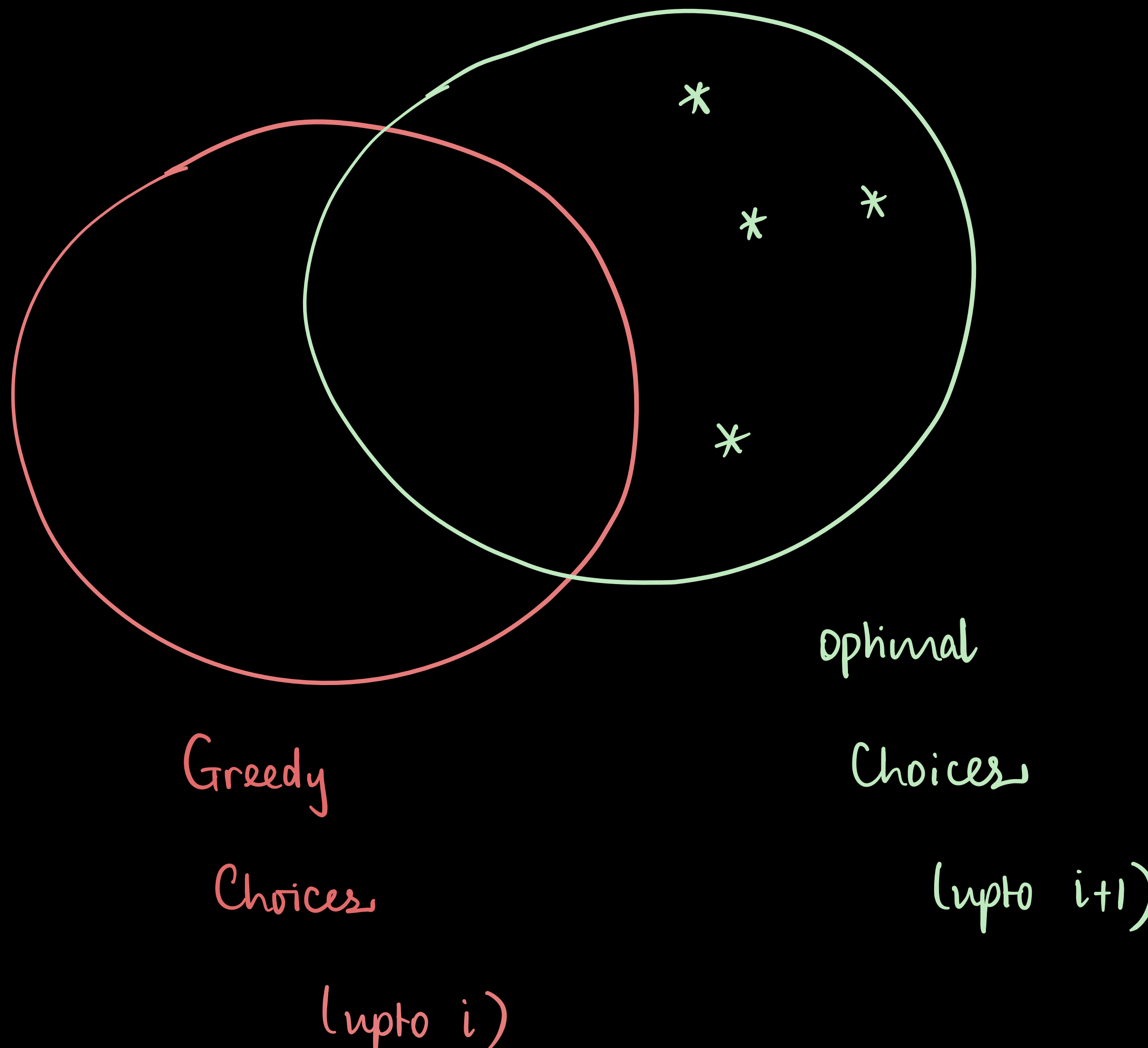
Greedy outcome : {★, o}   opt : {†, ~}

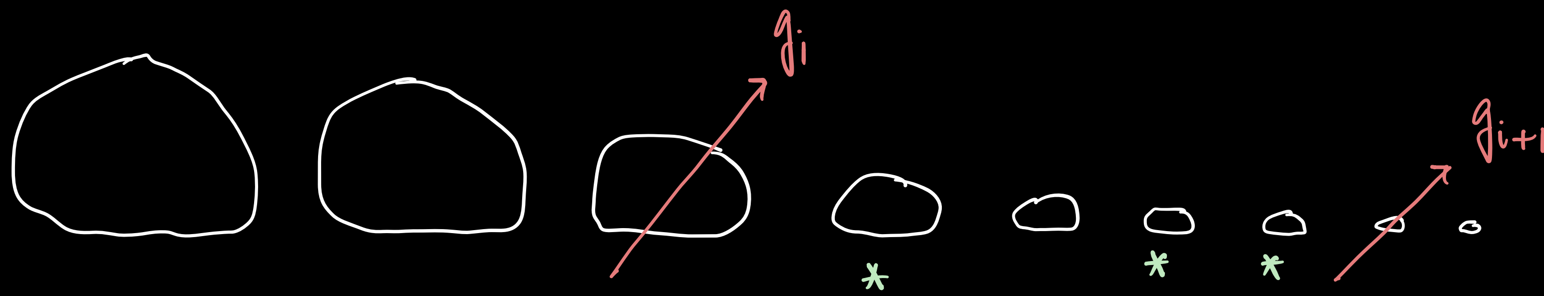
MAX edition

Line up the greedy choices & opt choices in DECREASING order of weight:

$$
\begin{array}{llllll}
\text{Greedy Choices} \rightarrow & g_1 & g_2 & \cdots & g_i & g_{i+1} \\
 & \geq & \geq & & \geq & < \\
\text{Optimal Choices} \rightarrow & h_1 & h_2 & \cdots & h_i & h_{i+1}
\end{array}
$$

Notice that <u>all</u> h_j's $(1 \leq j \leq i+1)$ are heavier than g_{i+1}. Why were none of them chosen by greedy?


Greedy Choices (upto i)
optimal Choices (upto i+1)



$$\underbrace{\{g_1, \dots, g_{i-1}\} \cup \{*\}}_{\notin \mathcal{I}}$$

$\forall$ elements $* \in$ opt $-$ greedy

Avoid this situation by definition!

A family $\mathcal{F}$ is <u>nice</u> if the following is true:

If $X, Y \in \mathcal{F}$ & $|Y| > |X|$,

then $\exists\, y \in Y \setminus X$ such that

$X \cup \{y\} \in \mathcal{F}$.

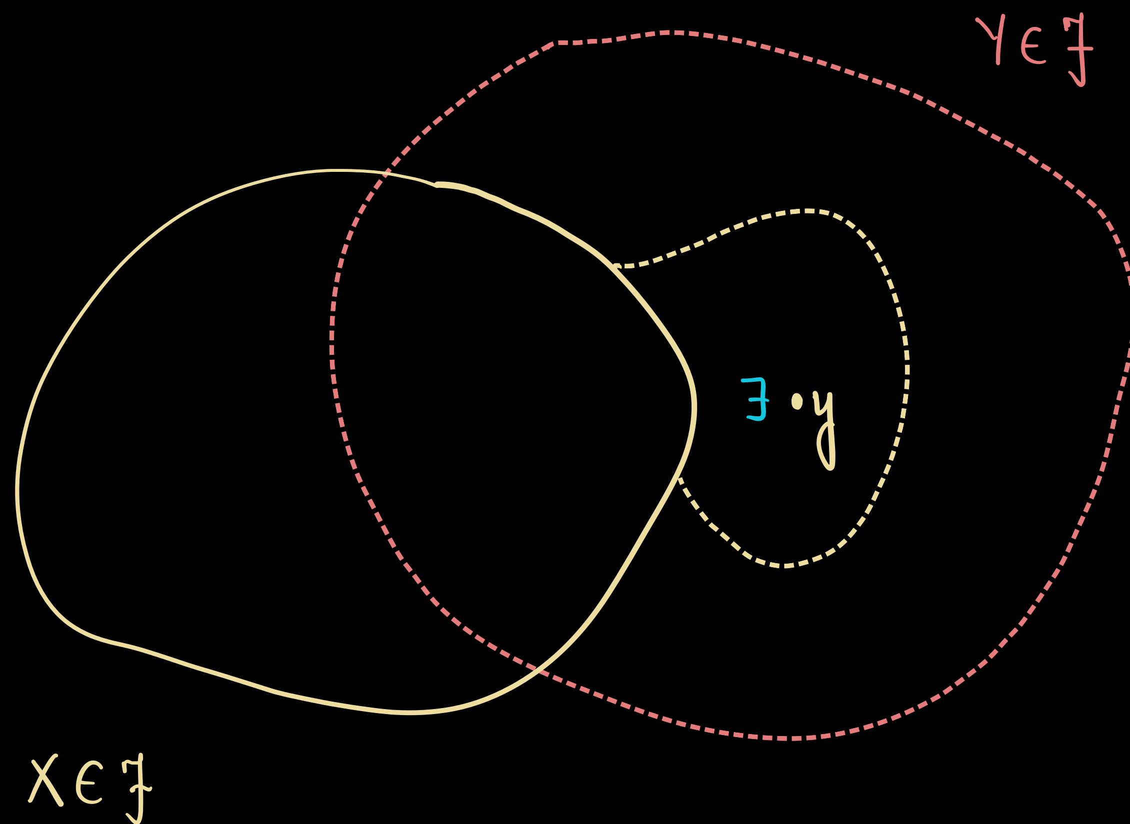
Avoid this situation by definition!

A family $\mathcal{F}$ is extra nice if the following is true:

If $X \in \mathcal{F}$, $Y \subseteq U$ & $|Y| > |X|$,

then $\exists\, y \in Y \setminus X$ such that

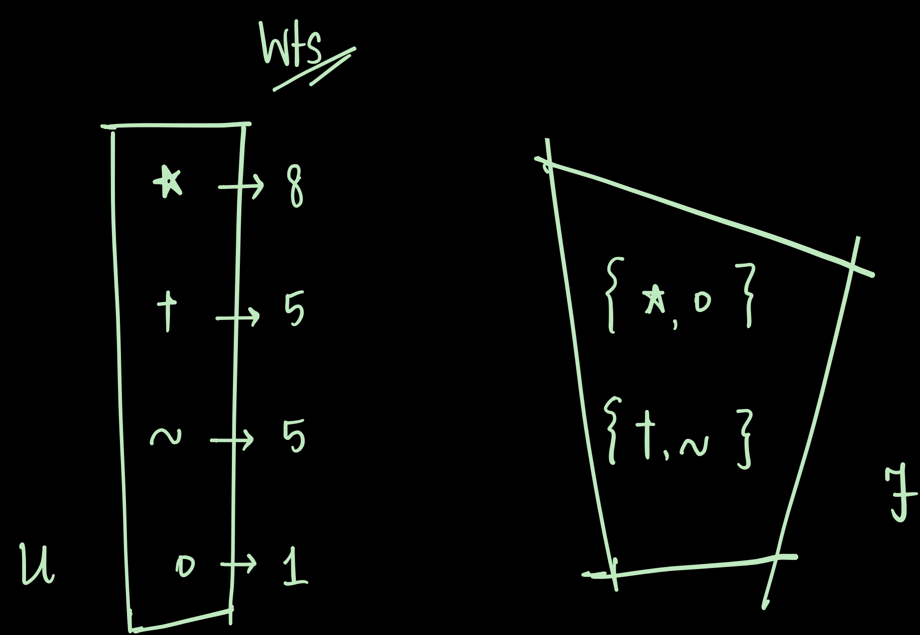
$X \cup \{y\} \in \mathcal{F}$.



If $|Y| > |X|$ then $\exists\, y \in Y \setminus X$ s.t. $X \cup \{y\} \in \mathcal{I}$.

Does the greedy algorithm work
if the input family is nice ?

Counter-example

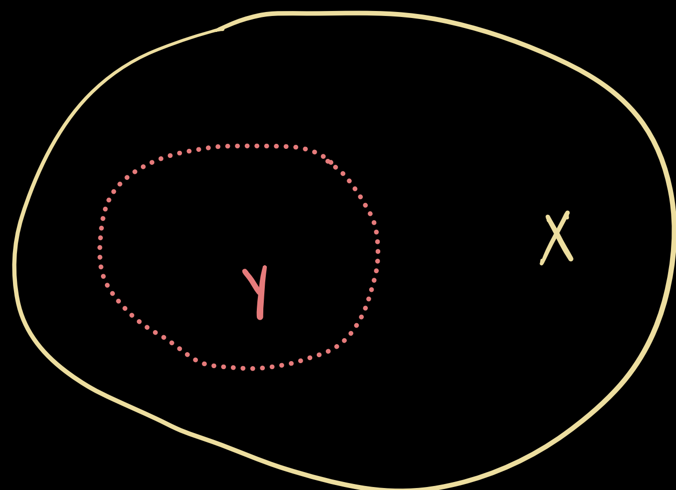


Greedy outcome : { }   opt : {†, ~}

Greedy Solution

built UP piece-by-piece

A family is super-nice if it is nice AND the following holds.



If $X \in \mathcal{F}$ & $Y \subseteq X$,

then $Y \in \mathcal{F}$ as well.

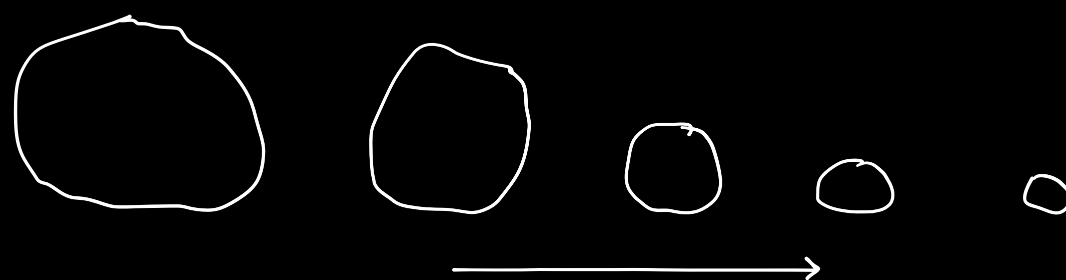
Does the greedy algorithm work

if the input family is super nice ?

GREEDY APPROACH.

$S = \emptyset$

Sort elements in decreasing order of weight.



For i in 1 to n:

    if $S \cup \{ i^{th}$ element $\} \in \mathcal{I}$

        add the i^{th} element to the solution

MAX edition

Line up the greedy choices & opt choices in DECREASING order of weight:

$$
\begin{array}{lcccccc}
\text{Greedy Choices} & \rightarrow & g_1 & g_2 & \cdots & g_i & g_{i+1} \\
 & & \geq & \geq & & \geq & < \\
\text{Optimal Choices} & \rightarrow & h_1 & h_2 & \cdots & h_i & h_{i+1}
\end{array}
$$

MAX edition

* non-empty

Line up the greedy choices & opt choices in DECREASING order of weight:

$$\begin{array}{llllll} \text{Greedy Choices} \rightarrow & g_1 & g_2 & \cdots & g_i & g_{i+1} \\ & \vee\!/ & \vee\!/ & & \vee\!/ & \wedge \\ \text{Optimal Choices} \rightarrow & h_1 & h_2 & \cdots & h_i & h_{i+1} \end{array}$$

MAX edition

Line up the greedy choices & opt choices in DECREASING order of weight:

$$
\begin{array}{lccccc}
\text{Greedy Choices} \rightarrow & g_1 & g_2 & \cdots & g_i & g_{i+1} \\
 & \vee\!/ & \vee\!/ & & \vee\!/ & \wedge \\
\text{Optimal Choices} \rightarrow & h_1 & h_2 & \cdots & h_i & h_{i+1}
\end{array}
$$

Define $X \rightsquigarrow \{g_1, \ldots, g_i\} \in \mathcal{I}$

$Y \rightsquigarrow \{h_1, \ldots, h_i, h_{i+1}\} \in \mathcal{I}$

$\exists\, 1 \leq j \leq i+1$ s.t. $X \cup \{h_j\} \in \mathcal{I}$ & $h_j \notin X$.

Since $\underbrace{wt(h_j) \geq wt(h_{i+1})}_{\text{we lined up the elements in decreasing order of weight}}$

& $\underbrace{wt(g_{i+1}) < wt(h_{i+1})}_{\text{by assumption}}$,

Also:

$h_j \notin g_l \ \forall\ 1 \leq l \leq i$

and

$\{g_1, \dots, g_i, h_j\} \in \mathcal{J}$.

we have : $wt(h_j) > wt(g_{i+1})$.

(Defn.) A matroid is a family $\mathcal{I}$ of subsets over an universe $\mathcal{U}$ that satisfies :

ⓐ Non-emptiness. $\emptyset \in \mathcal{I}$

ⓑ The Heriditary Property. If $Y \in \mathcal{I}$ & $X \subseteq Y$, then $X \in \mathcal{I}$.

ⓒ The Exchange Axiom. If $X, Y \in \mathcal{I}$ & $|Y| > |X|$, then $\exists\, y \in Y \setminus X$ s.t. $X \cup \{y\} \in \mathcal{I}$.

(Thm.) If $(\mathcal{U}, \mathcal{I})$ is a matroid and $\text{wt} : \mathcal{U} \to \mathbb{R}^+$ is a weight function on $\mathcal{U}$ and we define

$$\text{wt}(S) = \sum_{x \in S} \text{wt}(x)$$

for all $S \subseteq \mathcal{U}$, then the greedy algorithm returns a maximal set from $\mathcal{I}$ of maximum weight.