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Kernelization of the 3-path vertex cover problem

Christoph Brause *, Ingo Schiermeyer

Institute of Discrete Mathematics and Algebra, Technische Universität Bergakademie Freiberg, Prüferstraße 1, 09599 Freiberg, Germany

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ABSTRACT

The 3-path vertex cover problem is an extension of the well-known vertex cover problem. It asks for a vertex set $S \subseteq V(G)$ of minimum cardinality such that $G - S$ only contains independent vertices and edges. In this paper we will present a polynomial algorithm which computes two disjoint sets T_1, T_2 of vertices of G such that (i) for any 3-path vertex cover S' in $G[T_2]$, $S' \cup T_1$ is a 3-path vertex cover in G, (ii) there exists a minimum 3-path vertex cover in G which contains T_1 and (iii) $|T_2| \leq 6 \cdot \psi_3(G[T_2])$, where $\psi_3(G)$ is the cardinality of a minimum 3-path vertex cover and T_2 is the kernel of G.

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1. Introduction

During the last years the k-path vertex cover problem (k-PVCP for short) has become more and more interesting in graph theory since it is applicable to many practical problems. While there exist only a few results on k-PVCP, the number of open questions arises expeditiously. Motivated by the problem of ensuring data integrity of communication in wireless sensor networks using the k-generalized Canvas scheme in [6], Brešar et al. introduce the k-PVCP in [2].

A vertex subset $S \subseteq V(G)$ is a k-path vertex cover of G if $G - S$ contains no (not necessarily induced) path of length $k - 1$ in G. It is minimum if there exists no k-path vertex cover of smaller cardinality. We denote by $\psi_k(G)$ the cardinality of a minimum k-path vertex cover.

For $k = 2$, the k-PVCP is the well-known vertex cover problem (VCP for short), which is known to be NP-hard. Moreover, in [2] it is shown that the computation of $\psi_k(G)$ is NP-hard for $k \geq 3$.

Although the k-PVCP is NP-hard, there exist some approximation algorithms for $k \leq 3$. For example, using Nemhauser's and Trotter's result in [5], we have a factor-2 algorithm for $k = 2$. For larger k, it is widely unknown whether one can approximate the k-PVCP within a factor smaller than k. An exceptional case is $k = 3$, where two factor-2 algorithms are given by Tu and Zhou in [9] and [10].

For $k = 3$, one can find an approximation algorithm and some bounds for $\psi_3(G)$ in cubic graphs in [8], whereas [4] gives an exact algorithm to solve the 3-PVCP in time $\mathcal{O}^*(1.5171^n)$ for general graphs.

By Nemhauser's and Trotter's paper in 1975 [5], the question of finding the "hard part" of an NP-hard problem in a graph G arises. In that sense "hard part" means kernel, i.e. the remaining set of vertices after applying some polynomial reduction techniques. In [5], the authors deal with the VCP, i.e. 2-PVCP, and its kernel. Given d, the generalization of the VCP of Fellows et al. in [3] considers the problem of finding a vertex set of minimum cardinality whose removal from G yields a graph possessing vertices of degree at most d. Their result provides a polynomial algorithm computing a vertex set T such that the cardinality of an optimal solution is at most $|T|/(d^3 + 4d^2 + 6d + 4)$. It gives us a first kernelization algorithm for the 3-PVCP.

* Corresponding author.
E-mail addresses: brause@math.tu-freiberg.de (C. Brause), schiermeyer@math.tu-freiberg.de (I. Schiermeyer).

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On the one hand, we have two factor-2 approximation algorithms [9,10], which do not use kernelization techniques. On the other hand, we can compute a kernel $T \subseteq V(G)$ polynomially, such that $|T| \leq 15 \cdot \psi_3(G[T])$ (by [3] for $d = 1$). Since the range between 2 and 15 is really large, the aim of this paper is to provide a polynomial algorithm which computes a better kernel T for an arbitrary graph G, i.e. $|T| \leq 6 \cdot \psi_3(G[T])$.

We consider finite, simple and undirected graphs and use [1] for terminology and notation which are not defined here.

A vertex subset $S \subseteq V(G)$ is *independent* (*dissociative*) if $G[S]$ contains no P_2 (no P_3). A set of vertex disjoint P_3's is a P_3-*packing*. It is *maximal* if there exists no P_3 in G containing no vertex of a P_3 in the P_3-*packing*. For some maximal P_3-packing $\mathcal{P}$, the graph $G[V(G) \setminus V(\mathcal{P})]$ is the disjoint union of isolated vertices and isolated edges, i.e. $V(G) \setminus V(\mathcal{P})$ is a dissociative set in G. Let us denote by $\mathcal{P}_2(\mathcal{P})$ its set of isolated P_2's and by $\mathcal{P}_1(\mathcal{P})$ its set of isolated vertices. Furthermore, let us define $Q(\mathcal{P})$ as set of those vertices in $V(\mathcal{P})$ which have a neighbour in $V(\mathcal{P}_1(\mathcal{P})) \cup V(\mathcal{P}_2(\mathcal{P}))$. If P is a path in $\mathcal{P}$, then let us denote by $Q(P)$ the set of vertices in P which are in $Q(\mathcal{P})$. To simplify notation, let us say vertices in $V(\mathcal{P})$ are *black* and vertices in $V(\mathcal{P}_1(\mathcal{P})) \cup V(\mathcal{P}_2(\mathcal{P}))$ are *white* for some given $\mathcal{P}$. Moreover, all vertices in $Q(\mathcal{P})$ are called *contact* vertices. We define the white neighbourhood $N^w(u)$ of a black vertex u as the subset of white vertices which are either adjacent to u or have one white common neighbour with u. Additionally, let us define $N_1^w(u)$ and $N_2^w(u)$ as $N^w(u) \cap V(\mathcal{P}_1(\mathcal{P}))$ and $N^w(u) \cap V(\mathcal{P}_2(\mathcal{P}))$, respectively. Again to simplify notation, let us denote for some dissociative set D by $Q'(D)$ the set of all vertices $u \in N(D)$ for which every neighbour in D is not isolated in $G[D]$. To generalize our concepts, let us define by $f(T)$ the set $\bigcup_{u \in T} f(u) \setminus T$ for some function $f : V(G) \to 2^{V(G)}$ and some set $T \subseteq V(G)$.

2. Results

Our main objective is to provide a polynomial algorithm computing a kernel of the 3-path vertex cover problem in G. We need two important tools for it. First, we introduce the concept of a 3-path crown decomposition.

Definition. A 3-*path crown decomposition* (H, C, R) is a partition of the vertices of the graph G such that

(i) H (the header) separates C and R, i.e. there exist no edges between C and R,
(ii) C (the crown) is a dissociative set in G,
(iii) there exists a function $F : H \to \binom{C \cup H}{3}$ such that $\{G[F(u)] : u \in H\}$ is a P_3-packing in $G[H \cup C]$ of cardinality $|H|$ whose every path contains exactly one vertex of H.

Special cases of the 3-path crown decomposition are introduced by Prieto and Sloper in [7] and are known as *fat crown decomposition* and *double crown decomposition*. The first one requires the additional property that only end-vertices of the P_3's in the P_3-packing are elements of H while the second one considers C as an independent set in G.

The usefulness of the 3-path crown decomposition is presented in the next lemma.

Lemma 2.1. *A graph G that admits a* 3-*path crown decomposition* (H, C, R) *has a* 3-*path vertex cover of size at most c if and only if $G[R]$ has a* 3-*path vertex cover of size at most $c - |H|$.*

Proof. Since C is a dissociative set in G, $S' \cup H$ is a 3-path vertex cover of size $|S'| + |H|$ in G if $S' \subseteq R$ is a 3-path vertex cover in $G[R]$.

Let S be a 3-path vertex cover in G. Assume $H \cap (V(G) \setminus S) \neq \emptyset$. Due to the definition of a 3-path crown decomposition, we have a P_3-packing in $G[H \cup C]$ of cardinality $|H|$ where each P_3 has exactly one vertex in H. Since these P_3 are covered by H, it follows $|C \cap S| \geq |H \cap (V(G) \setminus S)|$. This inequality implies that $S \setminus (H \cup C)$ is a 3-path vertex cover of size at most $|S| - |H|$. □

According to the above lemma, $\psi_3(G) = \psi_3(G[R]) + |H|$ follows easily by deleting C and H of a 3-path crown decomposition. It indicates the importance of the lemma. Our aim is to provide a polynomial kernelization algorithm by using the concept of 3-path crown decomposition. The computation of the latter one can be divided into two steps.

The first one considers the fat crown decomposition.

Lemma 2.2 (*Prieto and Sloper [7]*). *Let G be a graph and $\mathcal{J}$ be a collection of independent P_2's such that $|\mathcal{J}| \geq |N(V(\mathcal{J}))|$. Then we can find a fat crown decomposition (H, C, R) where $C \subseteq V(\mathcal{J})$ and $H \subseteq N(V(\mathcal{J}))$ in linear time.*

Let G be a graph and D be a dissociative set. To find a 3-path crown decomposition, we contract all edges in D and obtain a graph G^*. By the above lemma, for some given dissociative set D in G either one can find a fat crown decomposition in linear time, which perhaps is a 3-path crown decomposition, or the number of contracted edges is bounded from above by $|N(D)|$. It helps us for the second step, which basically uses the property that we obtain an independent set by contracting all edges in D.

Lemma 2.3. *For a graph G and a dissociative set D, let G^* be the graph constructed by edge contraction in D and adding an additional vertex u', which is only adjacent to u, for every vertex $u \in Q'(D)$. Furthermore, let us denote by D^* the set $V(G^*) \setminus (V(G) \setminus D)$. If (H, C^*, R) is a double crown decomposition in G^* such that $C^* \subseteq D^*$ and $H \subseteq N(D^*)$, then $(H, V(G) \setminus (H \cup R), R)$ is a 3-path crown decomposition in G such that $V(G) \setminus (H \cup R) \subseteq D$ and $H \subseteq N(D)$.*

Proof. From the definition of G^* it is clear that $V(G) \setminus (H \cup R)$ is a dissociative set. Suppose that there exists an edge between $V(G) \setminus (H \cup R)$ and R in G. It implies the existence of an edge between C^* and R in G^*, which is a contradiction to the definition of the double crown decomposition (H, C^*, R) in G^*. Let us denote by F^* a function $F^* : H \to \binom{C \cup H}{3}$ such that $\{G[F^*(u)] : u \in H\}$ is a P_3-packing in $G[H \cup C^*]$ of cardinality $|H|$ whose every path contains exactly one vertex of H. For any $u \in H$, let us define $F[u]$ by $\{u, v_1, v_2\}$ if there exists a vertex $v \in F^*(u)$ which corresponds to the contracted edge v_1v_2 and $F^*[u]$ otherwise. Trivially, if the added vertex u' of $u \in Q(P')$ is an element of $F^*[u]$, then the third vertex in $F^*[u]$ corresponds to a contracted edge. Furthermore, by the definition of G^* and F^*, F fulfills all conditions in (iii) of the definition of a 3-path crown decomposition. $\square$

By the above lemma, it is clear that in the second step we reduce the problem of finding a 3-path crown decomposition in G to the problem of finding a double crown decomposition in G^*. We have the following lemma, proved by Prieto and Sloper in [7].

Lemma 2.4 (*Prieto and Sloper [7]*)**.** *Let G be a graph and I be an independent set such that $|I| \geq 2 \cdot |N(I)|$. We can find a double crown decomposition (H, C, R) such that $C \subseteq I$ and $H \subseteq N(I)$ in linear time.*

Combining both steps leads to the following lemma.

Lemma 2.5. *Let G be a graph and D be a dissociative set such that $|D| \geq 3 \cdot |N(D)| - |Q'(D)|$. We find a 3-path crown decomposition (H, C, R) such that $C \subseteq D$ and $H \subseteq N(D)$ in $\mathcal{O}(n^2)$.*

Proof. In $\mathcal{O}(n^2)$ we can decide whether or not the number of edges in D is at least $|N(D)|$ (complexity follows from the computation of $N(D)$). In the first case, we find a fat crown decomposition by Lemma 2.2 in linear time. In the second case, we contract at most $|N(D)|$ edges in $\mathcal{O}(m)$. Furthermore, we can compute $Q'(D)$ by checking for each vertex whether or not it is an element of $V(G) \setminus D$ or is isolated in $G[D]$ or none of both, and secondly considering all vertices in $V(G) \setminus D$ and check whether or not they have a neighbour which is isolated in $G[D]$ in $\mathcal{O}(n^2)$. Afterwards, adding for every $u \in Q'(D)$ a vertex u', which is only adjacent to u, can be done in $\mathcal{O}(n)$ (please note that the number of new vertices is at most n). Altogether gives a complexity of $\mathcal{O}(n^2)$ for the computation of the new independent set D^* and its neighbourhood $N(D^*) = N(D)$. The cardinality of D^* is the number of vertices in D plus the number of added vertices minus the number of contracted edges. Therefore, $|D| \geq 2 \cdot |N(D)|$. Now we find a 3-path crown decomposition by Lemma 2.4 in linear time. $\square$

With Lemma 2.5 we have a powerful decomposition tool. Next, we focus on the computation of a special maximal P_3-packing, which is the second main tool we need to design our algorithm.

As further notation, let us define for some P_3-packing $\mathcal{P}$ the set $\mathcal{T}_{(iii)}(\mathcal{P})$ as follows: Add all P_3's of $\mathcal{P}$ which have in the currently considered graph at most 3 white neighbours to $\mathcal{T}_{(iii)}(\mathcal{P})$, delete $N_G[V(\mathcal{T}_{(iii)}(\mathcal{P}))]$, and repeat this process as long as such P_3's exist in $\mathcal{P}$.

Now by deleting all paths in $\mathcal{T}_{(iii)}(\mathcal{P})$ and their white neighbours we may obtain new white neighbourhoods for paths not in $\mathcal{T}_{(iii)}(\mathcal{P})$. Note that the number of white neighbours is at least 4 and the number of contact vertices on a path may reduce. Using the new white neighbourhoods, let $\mathcal{T}_{(iv)}(\mathcal{P})$ be the set of P_3's in $\mathcal{P}$ with one contact vertex and at least 4 white neighbours, and $\mathcal{T}_{(v)}(\mathcal{P})$ be the set of P_3's in $\mathcal{P}$ which are not in $\mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(iv)}(\mathcal{P})$ but fulfill $|N_1^w(\{v_1, v_3\})| = 1$, $N_1^w(v_2) = \emptyset$, and $N_2^w(\{v_1, v_3\}) = \emptyset$.

Lemma 2.6. *Let $\mathcal{P}$ be a P_3-packing, and $P = v_1v_2v_3$ be a path of length 2 in $\mathcal{P}$. Then at least one of the following statements is true:*

(i) *we find a P_3-packing $\mathcal{P}'$ of cardinality at least $|\mathcal{P}| + 1$,*
(ii) *we find a P_3-packing $\mathcal{P}'$ of cardinality $|\mathcal{P}|$ such that $|\mathcal{P}_2(\mathcal{P}')| > |\mathcal{P}_2(\mathcal{P})|$,*
(iii) *$P \in \mathcal{T}_{(iii)}(\mathcal{P})$,*
(iv) *$P \in \mathcal{T}_{(iv)}(\mathcal{P})$,*
(v) *$P \in \mathcal{T}_{(v)}(\mathcal{P})$.*

Proof. Suppose none of the statements is true. Since P is not an element of $\mathcal{T}_{(iii)}(\mathcal{P})$, we consider the graph obtained by deleting all paths of $\mathcal{T}_{(iii)}(\mathcal{P})$ and their white neighbours. Now, every path in $\mathcal{P}$ has at least four white neighbours in the obtained graph. Since P is not a path in $\mathcal{T}_{(iv)}(\mathcal{P})$, P has at least 2 contact vertices.

Assume $N_2^w(P) \neq \emptyset$. Then there exists a path of two vertices, denoted by w_1 and w_2, in $G[N^w(P)]$. We assume, without loss of generality, that w_1 is adjacent to a vertex in $Q(P)$.

If w_1 is adjacent to v_1 or v_3, without loss of generality, assume it is v_1, and a white vertex w_3 of $N^w(P) \setminus \{w_1, w_2\}$ has a neighbour in $\{v_2, v_3\}$, then we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{G[\{v_1, w_1, w_2\}], G[\{v_2, v_3, w_3\}]\}$ with the property described in (i). By this contradiction, we can assume, without loss of generality, that w_1 is adjacent to v_1 and all vertices in $N^w(P) \setminus \{w_1, w_2\}$ are non-adjacent to v_2 and v_3. Consequently, by $N^w(P) \geq 4$, there exist two distinct vertices $w_3, w_4 \in N^w(P) \setminus \{w_1, w_2\}$ such that $G[\{v_1, w_3, w_4\}]$ is a P_3. By $|Q(P)| \geq 2$ we have that w_1 or w_2 is adjacent to v_2 or v_3 (say w_i is adjacent to one of them), which implies that we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{G[\{v_1, w_3, w_4\}], G[\{v_2, v_3, w_i\}]\}$ with the property described in (i), a contradiction.

The above contradictions imply that no vertex in $N_2^w(P)$ is adjacent to v_1 or v_3.

Let us consider the case where w_1, the end-vertex of a white path of two vertices, is adjacent to v_2. Let us denote by w_2 its neighbour in $N_2^w(P)$. Since we assume $|Q(P)| \geq 2$, there exists at least one vertex $w_3 \in N_1^w(P)$ which is adjacent to v_1 or v_3. Assume, without loss of generality, that w_3 is a neighbour of v_1. If there exists a vertex $w_4 \in N_1^w(P)$, different from w_1, w_2 and w_3, then we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{w_3v_1w_4, v_2w_1w_2\}$ if w_4 is adjacent to v_1, a contradiction to (i). Hence, w_4 is non-adjacent to v_1. Now, $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{v_3v_2w_4\}$ is a P_3-packing if w_4 is adjacent to v_2. This contradicts (ii) and implies that w_4 is adjacent to v_3. Therefore, we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{w_3v_1v_2, w_4v_3w_5\}$ or $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{v_2v_3w_4\}$ depending on whether or not there exists a vertex $w_5 \in N_w^1(v_3) \setminus \{w_1, w_2, w_3, w_4\}$. This contradicts (i) or (ii). Now (v) follows, a contradiction.

Note that, above transformations imply $|\mathcal{P}_2(\mathcal{P})| \leq |\mathcal{P}_2(\mathcal{P}')| + 2$ whenever $|\mathcal{P}'| > |\mathcal{P}|$.

The above contradictions imply $N_2^w(P) = \emptyset$ and $|N_1^w(P)| \geq 4$. If $|N_1^w(v_1)| \geq 2$ or $|N_1^w(v_3)| \geq 2$, then, without loss of generality, let $w_1, w_2 \in N_1^w(v_1)$. Since $|Q(P)| \geq 2$, without loss of generality, there exists a vertex $w_3 \in N_1^w(P) \setminus \{w_1, w_2\}$ which is adjacent to v_2 or v_3 and we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{G[\{v_1, w_1, w_2\}], G[\{v_2, v_3, w_3\}]\}$ with the property described in (i). This contradiction implies that we can assume, without loss of generality, that $|N_1^w(v_1)| = 1$ and $|N_1^w(v_2) \setminus N_1^w(v_1)| \geq 2$, i.e. let $w_1 \in N_1^w(v_1)$, $w_2, w_3 \in N_1^w(v_2)$ be pairwise distinct vertices. Now we find a P_3-packing $\mathcal{P}' = (\mathcal{P} \setminus \{v_1v_2v_3\}) \cup \{w_2v_2w_3\}$ with the property described in (ii), a contradiction.

The above contradictions prove the lemma. □

The proof of the above lemma consists of a case distinction on $N^w(P)$. For some P_3-packing, we can compute $N_1^w(u)$ and $N_2^w(u)$ for all black vertices $u \in V(G)$ based on the following algorithm in $\mathcal{O}(n^2)$: For each vertex check whether it is black or it is white and has no white neighbour or it is white and has a white neighbour. Afterwards, for every black vertex u and every white neighbour v of u, v belongs to $N_1^w(u)$ if v is white and has no white neighbour. Otherwise, v and its white neighbour belong to $N_2^w(u)$.

After computing the white neighbourhood, we can check whether or not a path of $\mathcal{P}$ has at most 3 white neighbours, in a positive case, add it to $\mathcal{T}_{(iii)}(\mathcal{P})$, delete all of its neighbours from the white neighbourhoods of other paths in $\mathcal{O}(n)$, and repeat this process as long as possible. Since the number of paths is at most $\mathcal{O}(n)$ and we have at most $\mathcal{O}(n)$ repetitions, this procedure can be done in at most $\mathcal{O}(n^3)$.

Now the decision on whether or not (iii) or (iv) of Lemma 2.6 is fulfilled and, in a negative case, the computation of $\mathcal{P}'$ can be done in $\mathcal{O}(1)$ for some given path $P \in \mathcal{P} \setminus \mathcal{T}_{(iii)}(\mathcal{P})$ and in $\mathcal{O}(n)$ for all paths of $\mathcal{P} \setminus \mathcal{T}_{(iii)}(\mathcal{P})$.

The inductive repetition of the operations in the proof of Lemma 2.6 provides a method, how to compute a maximal P_3-packing $\mathcal{P}'$ such that $\mathcal{P}' = \mathcal{T}_{(iii)}(\mathcal{P}') \cup \mathcal{T}_{(iv)}(\mathcal{P}') \cup \mathcal{T}_{(v)}(\mathcal{P}')$.

Let us denote the transition of $\mathcal{P}$ to $\mathcal{P}'$ as suggested in statements (i) and (ii) of Lemma 2.6 as an *operation*. It is clear that at least the cardinality of the P_3-packing or the number of P_2's in the remaining graph increases. But by further observation on the case analysis in the proof, whenever the number of P_3's increases, the cardinality of $\mathcal{P}_2(\mathcal{P}')$ decreases by at most 2. Hence, we apply at most n operations to find a maximal P_3-packing $\mathcal{P}'$ such that $\mathcal{P}' = \mathcal{T}_{(iii)}(\mathcal{P}') \cup \mathcal{T}_{(iv)}(\mathcal{P}') \cup \mathcal{T}_{(v)}(\mathcal{P}')$.

The following idea computes a maximal P_3-packing in $\mathcal{O}(m \cdot n)$. Let $\mathcal{P}$ be an empty P_3-packing. For every edge $uv \in E(G)$, check whether or not u and v are in $V(\mathcal{P})$. If both are not, then check whether or not there exists a neighbour $w \in V(G) \setminus V(\mathcal{P})$ of $\{u, v\}$. In a positive case, add $G[\{u, v, w\}]$ to $\mathcal{P}$ and continue with the next edge.

The above statements imply the following lemma.

Lemma 2.7. *One can compute a maximal P_3-packing $\mathcal{P}$ such that $\mathcal{P} = \mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})$ in $\mathcal{O}(n^4)$.*

We have introduced the two main tools of our proof. Let us consider a P_3-packing $\mathcal{P}$ as stated in Lemma 2.7. Then we denote by $S_2(\mathcal{P})$ the set $V(\mathcal{P}_2(\mathcal{P})) \setminus N^w(V(\mathcal{T}_{(iii)}(\mathcal{P})))$, i.e. it is the vertex set of all P_2's in $\mathcal{P}_2(\mathcal{P})$, which are not in the white neighbourhood of a P_3 in $\mathcal{T}_{(iii)}(\mathcal{P})$. Similarly, $S_1(\mathcal{P})$ is defined as the set $V(\mathcal{P}_1(\mathcal{P})) \setminus N^w(V(\mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})))$, i.e. it is the set of all vertices in $\mathcal{P}_1(\mathcal{P})$, which are not in the white neighbourhood of a P_3 in $\mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})$. $S_1'(\mathcal{P})$ denotes the set $N_1^w(V(\mathcal{T}_{(v)}(\mathcal{P}))) \setminus N^w(V(\mathcal{T}_{(iii)}(\mathcal{P})))$, i.e. all vertices of $\mathcal{P}_1(\mathcal{P})$, which are adjacent to some P_3 in $\mathcal{T}_{(v)}(\mathcal{P})$ but not to some P_3 in $\mathcal{T}_{(iii)}(\mathcal{P})$. Clearly, $S_2(\mathcal{P}), S_1(\mathcal{P}), S_1'(\mathcal{P}), N^w(V(\mathcal{T}_{(iii)}(\mathcal{P})))$ is a decomposition of $V(G) \setminus V(\mathcal{P})$.

Please recall that $N_1^w(v)$ and $N_2^w(v)$ are computable for all $v \in V(G) \setminus V(\mathcal{P})$ in $\mathcal{O}(n^2)$. Afterwards, we can compute $\mathcal{T}_{(iii)}(\mathcal{P}), \mathcal{T}_{(iv)}(\mathcal{P})$ and $\mathcal{T}_{(v)}(\mathcal{P})$ in $\mathcal{O}(n^3)$. Since the white neighbourhood of a vertex is already computed, we find $N^w(\mathcal{T}_{(iii)}(\mathcal{P}))$ by selecting each white neighbour of a vertex in $V(\mathcal{T}_{(iii)}(\mathcal{P}))$ in $\mathcal{O}(n)$. Now any white vertex in $N_1^w(V(\mathcal{P}))$, which was not selected by the previous step, belongs to $S_1'(\mathcal{P})$ or $S_1(\mathcal{P})$ depending on whether or not it is adjacent to a vertex in $\mathcal{T}_{(v)}(\mathcal{P})$. Trivially, this can be now decided in $\mathcal{O}(n)$. Similarly, we have the same time complexity for computing $S_2(\mathcal{P})$.

Using above notations and complexities, we obtain Algorithm 1 and our main theorem.

Theorem 2.8. *Algorithm 1 computes two disjoint sets T_1, T_2 in $\mathcal{O}(n^5)$ such that*

(i) *for any 3-path vertex cover S' in $G[T_2]$, $S' \cup T_1$ is a 3-path vertex cover in G,*
(ii) *there exists a minimum 3-path vertex cover in G which contains T_1, and*
(iii) $|T_2| \leq 6 \cdot \psi_3(G[T_2])$.

Algorithm 1

1: $T_1 := \emptyset, T_2 := V(G)$
2: Compute a maximal P_3-packing $\mathcal{P}$ in G such that $\mathcal{P} = \mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})$.
3: **while** $|S_2(\mathcal{P}) \cup S_1(\mathcal{P})| \geq 3 \cdot |N(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))| - |Q'(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))|$ **do**
4: Compute a 3-path crown decomposition (H, C, R) in $G[T_2]$ using the dissociative set $S_2(\mathcal{P}) \cup S_1(\mathcal{P})$.
5: $T_1 := T_1 \cup H, T_2 := T_2 \setminus (H \cup C)$
6: Compute a maximal P_3-packing $\mathcal{P}$ in G such that $\mathcal{P} = \mathcal{T}_{(iii)}(\mathcal{P}) \cup \mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})$.
7: **end while**

Proof. The time-complexity for steps 2, 4, 6 and the decision in 3 is given by the above results. Moreover, since we delete vertices, the number of loop repetitions is at most n. This observation gives the complexity of our algorithm. Obviously, T_2 is the set of vertices which remain in the graph after the algorithm stops. Furthermore, T_1 consists of all headers H. The concept of 3-path crown decomposition gives (i) and (ii).

Let $\mathcal{P}$ be a maximal P_3-packing computed in steps 2 or 6 for $G[T_2]$. Then $\psi_3(G[T_2]) \geq |\mathcal{P}|$. Moreover, by definition we have

$$T_2 = \left[V(\mathcal{T}_{(iii)}(\mathcal{P})) \cup N^w(V(\mathcal{T}_{(iii)}(\mathcal{P})))\right] \cup \left[[V(\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P}))] \cup S_2(\mathcal{P}) \cup S_1(\mathcal{P}) \cup S_1'(\mathcal{P})\right]$$

and

$$\left|V(\mathcal{T}_{(iii)}(\mathcal{P})) \cup N^w(V(\mathcal{T}_{(iii)}(\mathcal{P})))\right| \leq 6 \cdot \left|\mathcal{T}_{(iii)}(\mathcal{P})\right|.$$

Since every path in $\mathcal{T}_{(v)}(\mathcal{P})$ has at most one white neighbour in $S_1'(\mathcal{P})$, we have $|S_1'(\mathcal{P})| \leq |\mathcal{T}_{(v)}(\mathcal{P})|$. Furthermore, every path in $\mathcal{T}_{(v)}(\mathcal{P})$ contains a vertex of $Q'(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))$. Hence, it follows $|S_1'(\mathcal{P})| \leq |Q'(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))|$. We conclude

$$\begin{aligned}&\left|\left[V(\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P}))\right] \cup S_2(\mathcal{P}) \cup S_1(\mathcal{P}) \cup S_1'(\mathcal{P})\right| \\ &\quad \leq \left|\left[V(\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P}))\right] \cup S_2(\mathcal{P}) \cup S_1(\mathcal{P})\right| + \left|Q'(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))\right|.\end{aligned}$$

Since the condition for applying the loop is not fulfilled and every path in $\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})$ contains at most one vertex which is adjacent to vertices in $S_2(\mathcal{P}) \cup S_1(\mathcal{P})$,

$$\left|\left[V(\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P}))\right] \cup S_2(\mathcal{P}) \cup S_1(\mathcal{P})\right| \leq 6 \cdot \left|\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})\right| - \left|Q'(S_2(\mathcal{P}) \cup S_1(\mathcal{P}))\right|$$

is true. Together, both inequalities imply

$$\left|\left[V(\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P}))\right] \cup S_2(\mathcal{P}) \cup S_1(\mathcal{P}) \cup S_1'(\mathcal{P})\right| \leq 6 \cdot |\mathcal{T}_{(iv)}(\mathcal{P}) \cup \mathcal{T}_{(v)}(\mathcal{P})|.$$

Now we obtain the desired result $|T_2| \leq 6 \cdot |\mathcal{P}| \leq 6 \cdot \psi_3(G[T_2])$. $\square$

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