Matroid Intersection

Input: Two matroids $\underbrace{\mathcal{M}_1 = (X, \mathcal{I}_1) \text{ and } \mathcal{M}_2 = (X}_{}, \mathcal{I}_2)$.

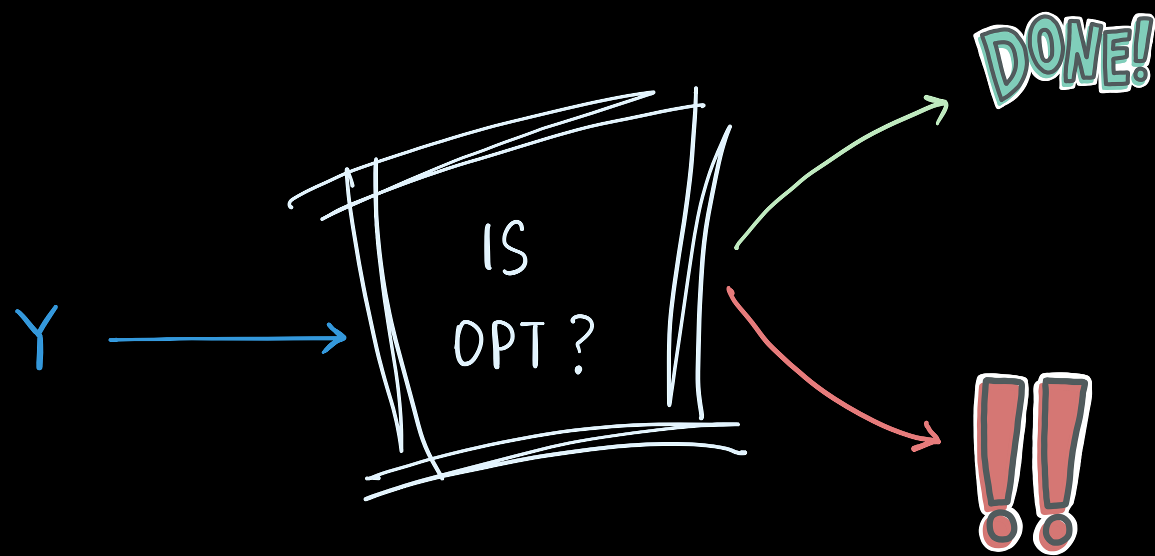
note that the ground set
is the same for both $\mathcal{M}_1$ & $\mathcal{M}_2$

GOAL. Find a subset $Y \subseteq X$ such that

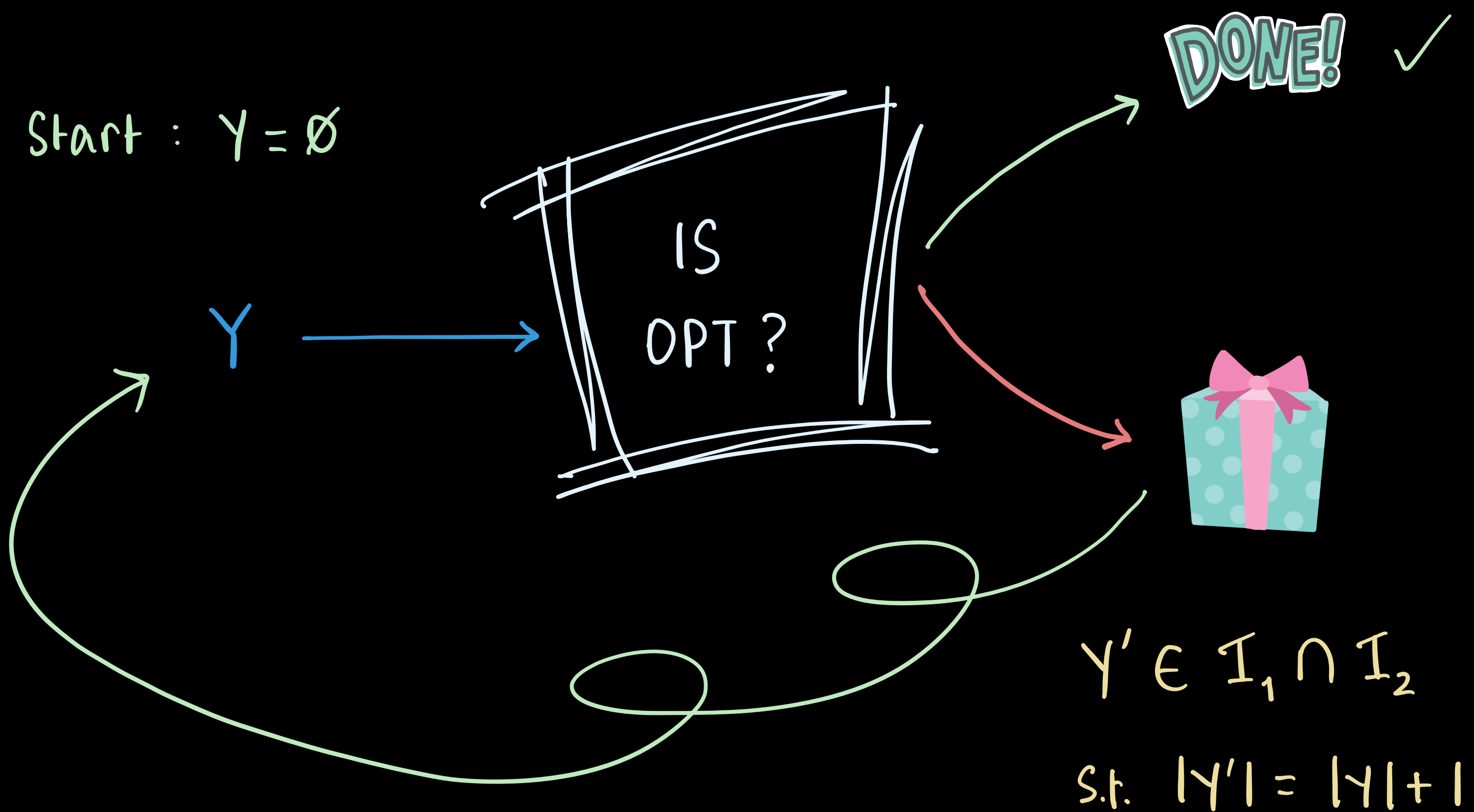
$$Y \in \mathcal{I}_1 \cap \mathcal{I}_2$$

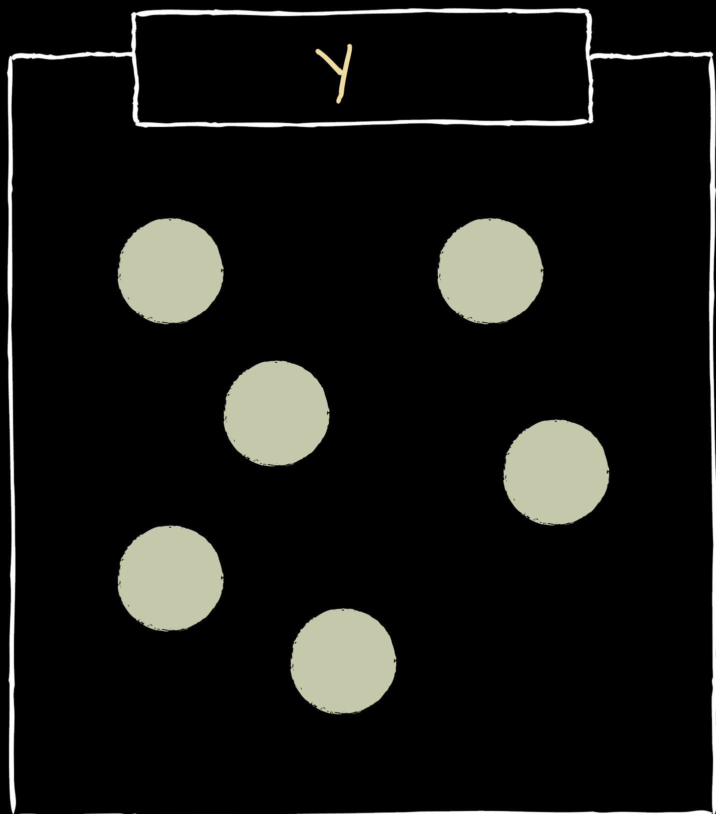
and $|Y|$ is maximized.

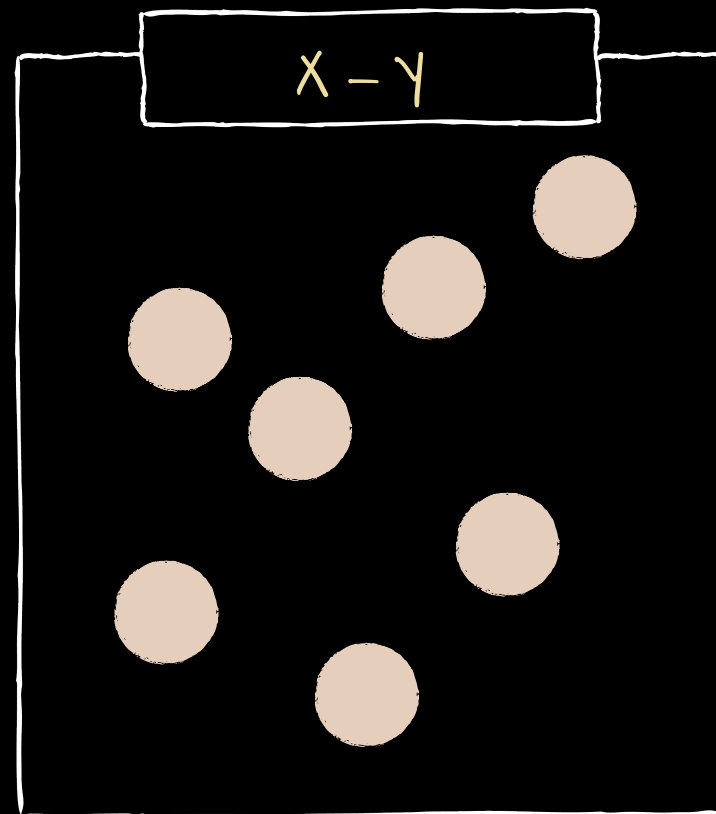
Idea: Develop an optimality detector.

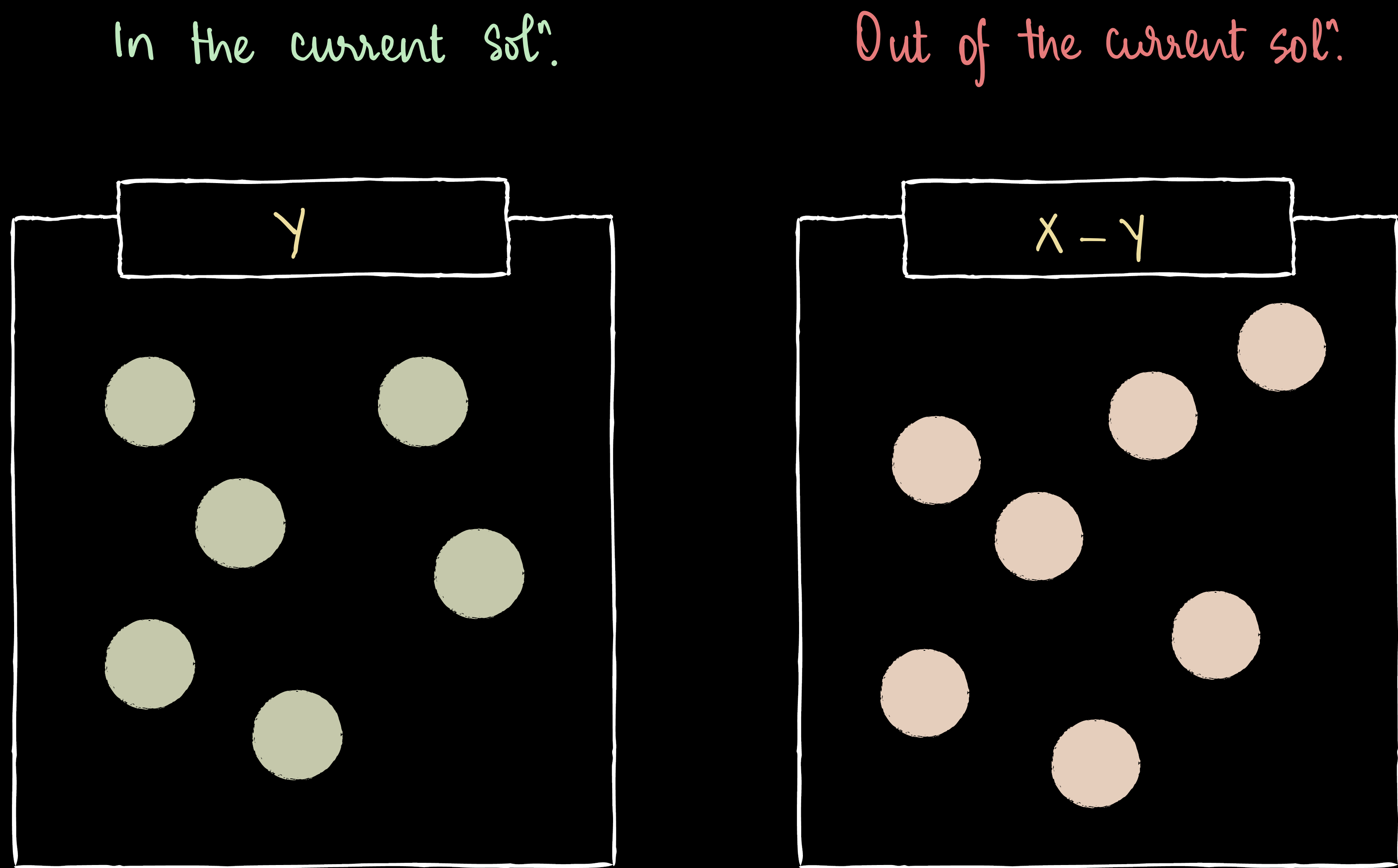


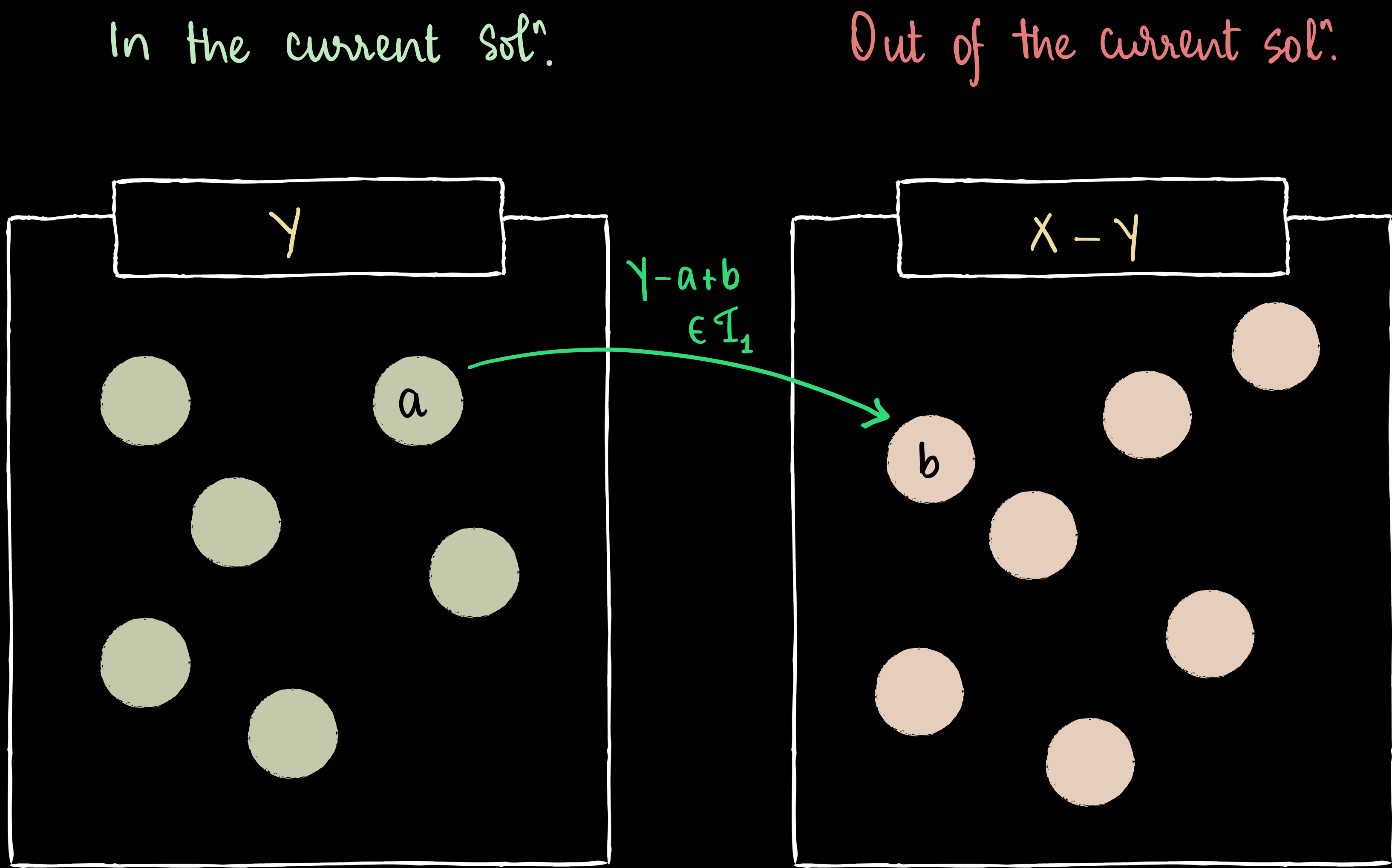
Idea: Develop an optimality detector.

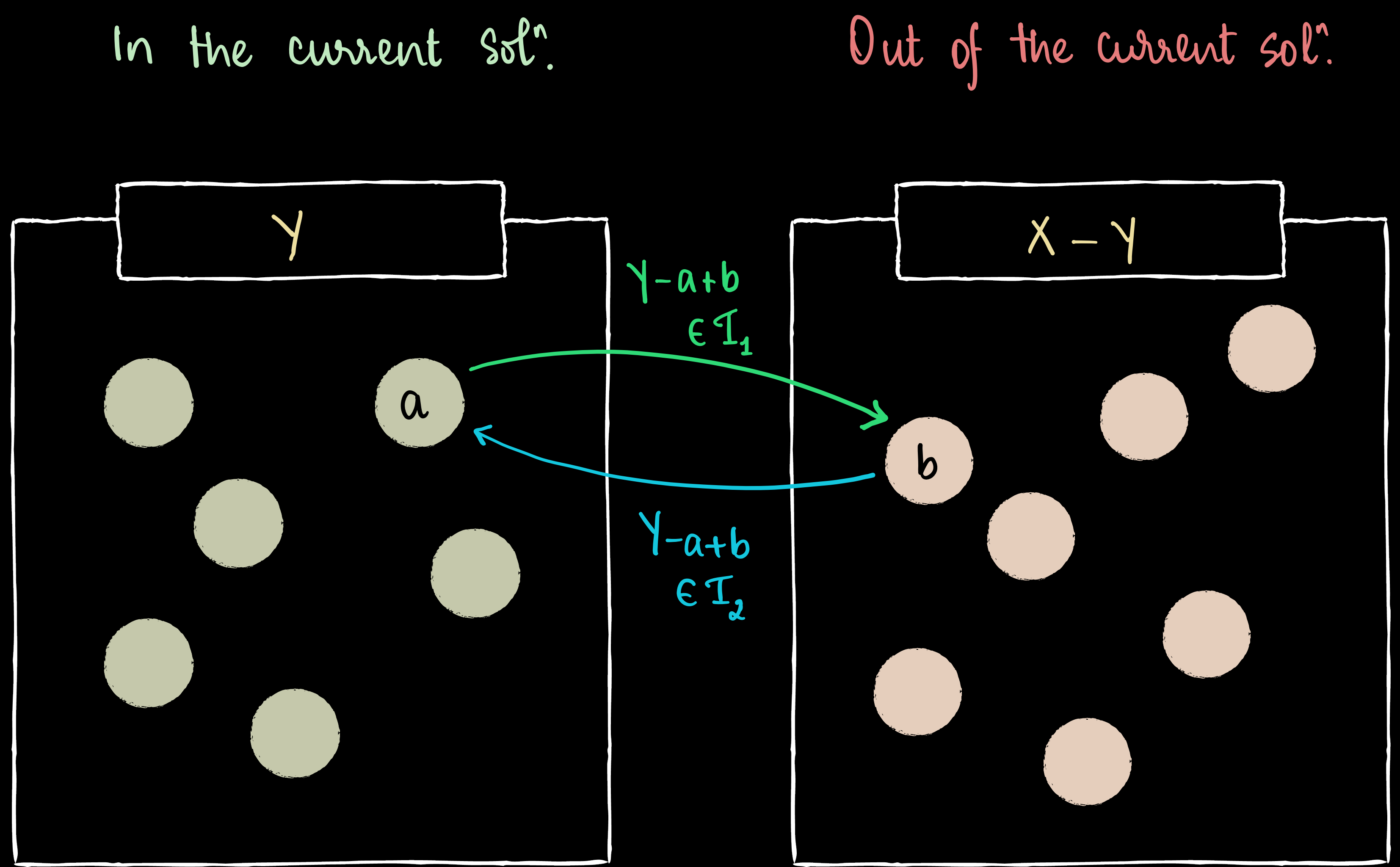



Y

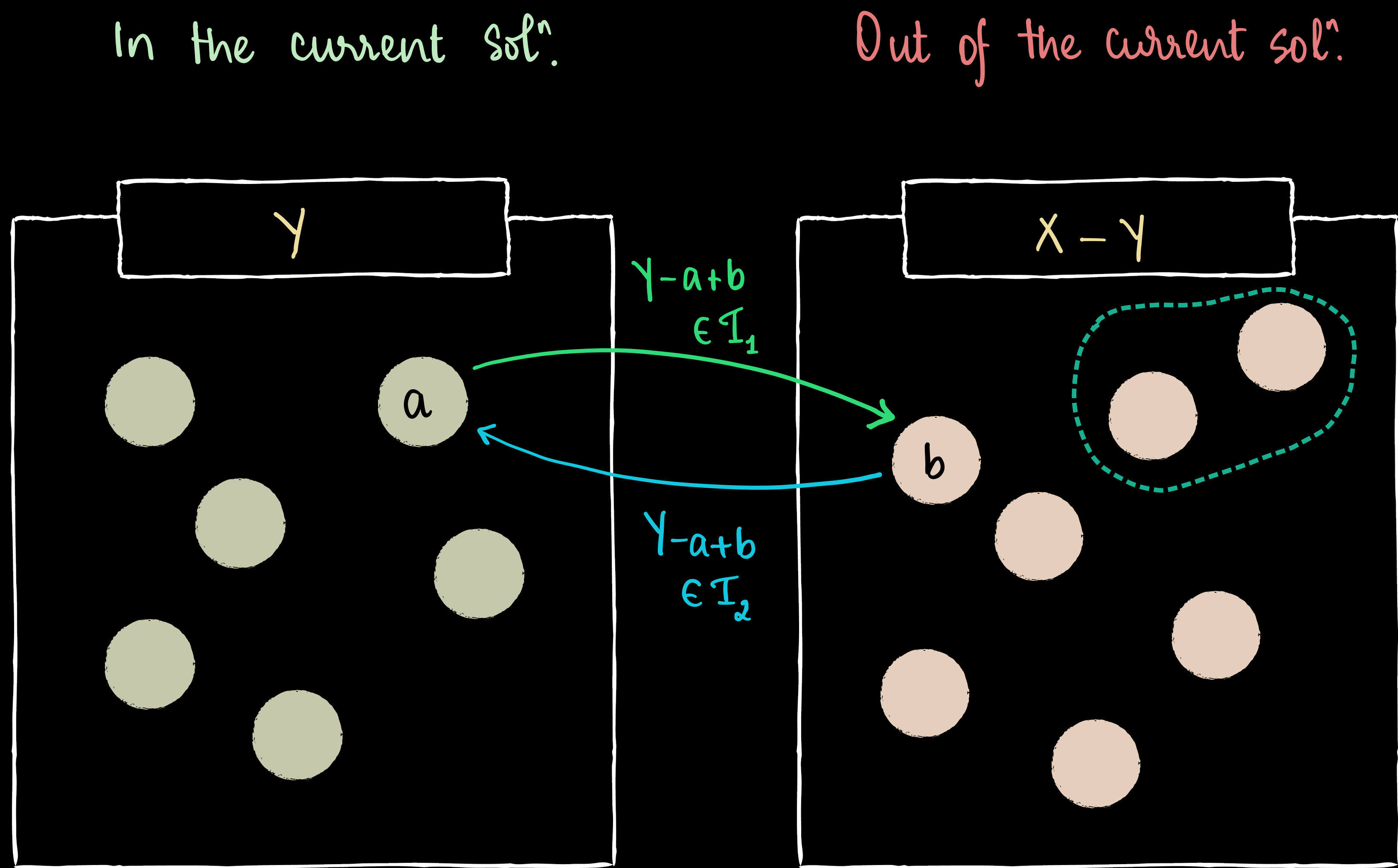

X − Y


In the current sol^n.
Out of the current sol^n.
Y
$X - Y$

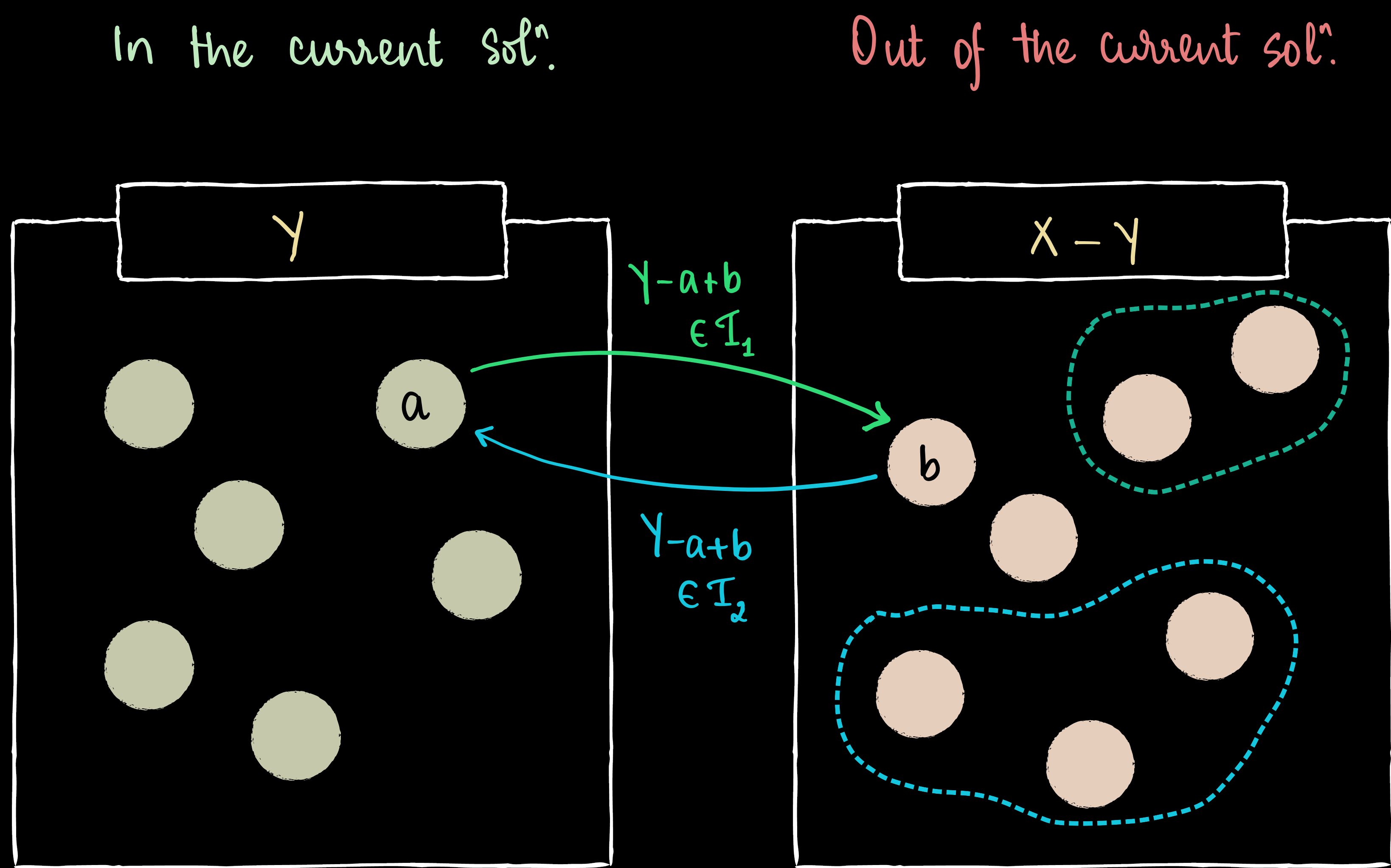

In the current sol^{n}.
Out of the current sol^{n}.
Y
$X - Y$
$Y - a + b$
$\in \mathcal{I}_1$
a
b



In the current sol^n.
Out of the current sol^n.
Y
X − Y
a
b
Y−a+b ∈ I_1
Y−a+b ∈ I_2



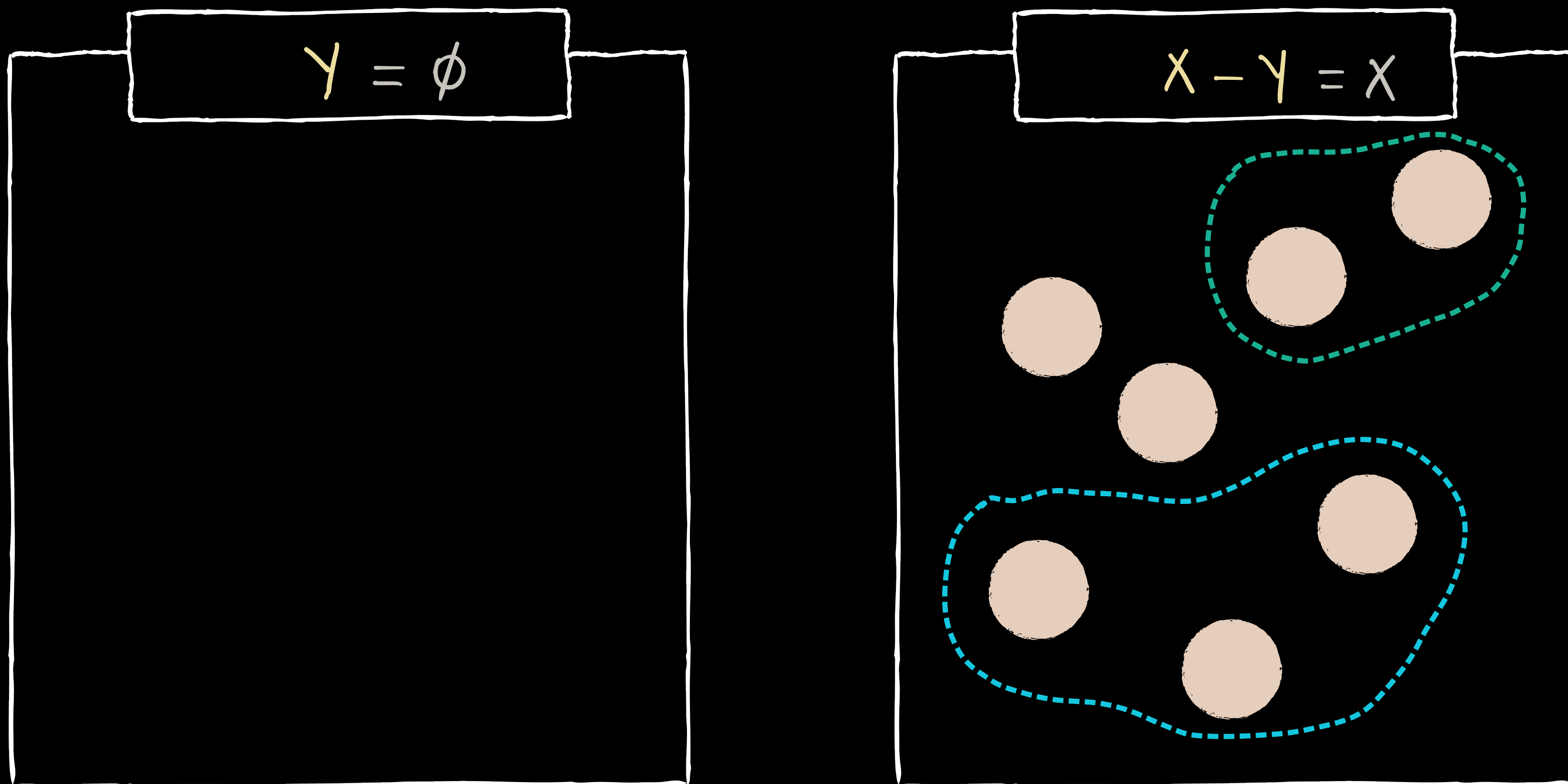
$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

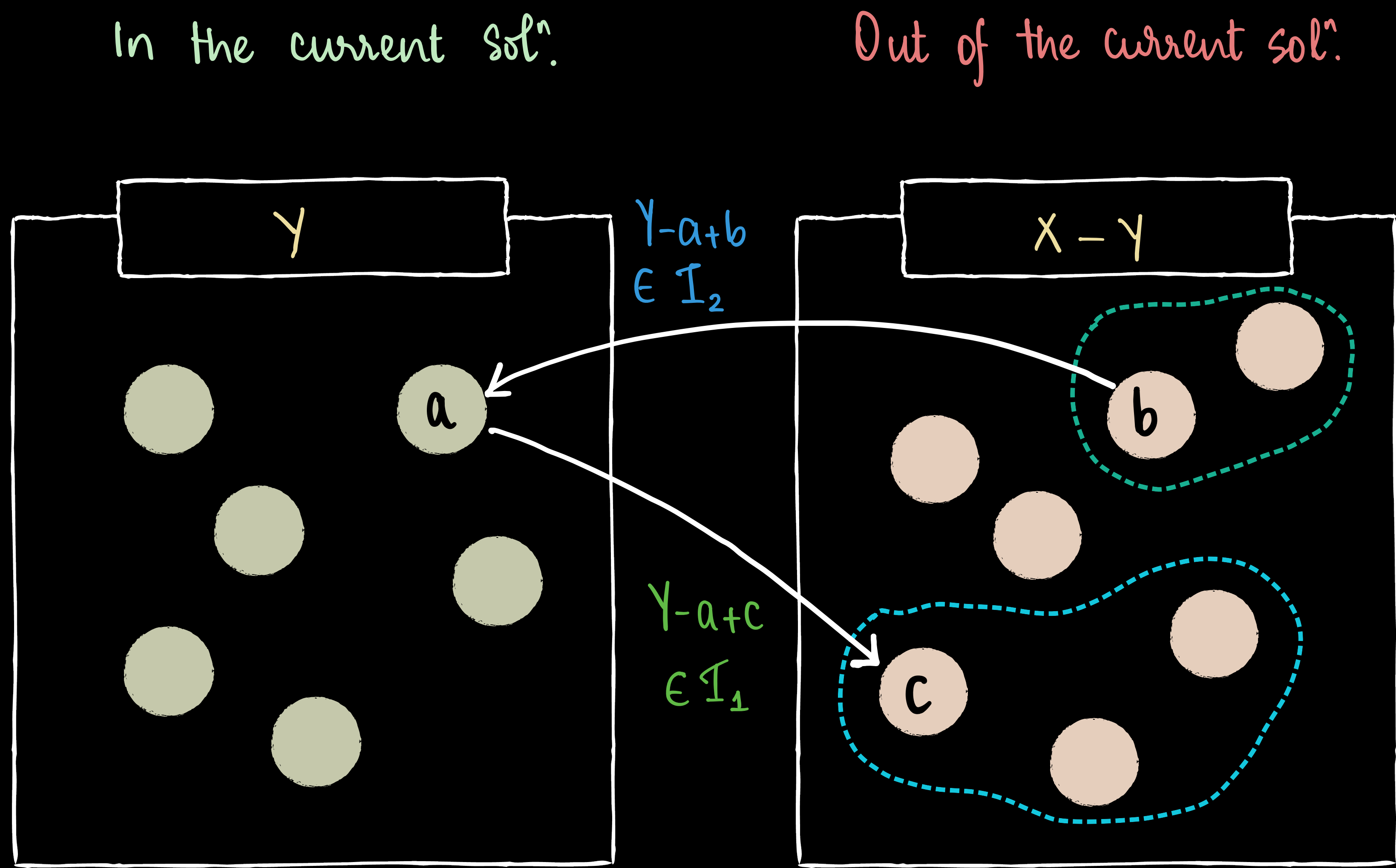
$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

In the current sol^n. Out of the current sol^n.



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$



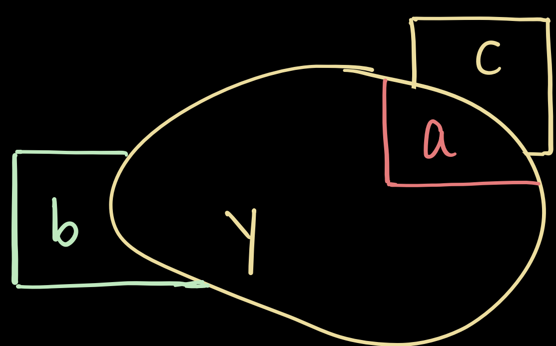
$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

What can we say about

$$Y \cup \{b,c\} - \{a\} ?$$

extras = $\{a, b\}$



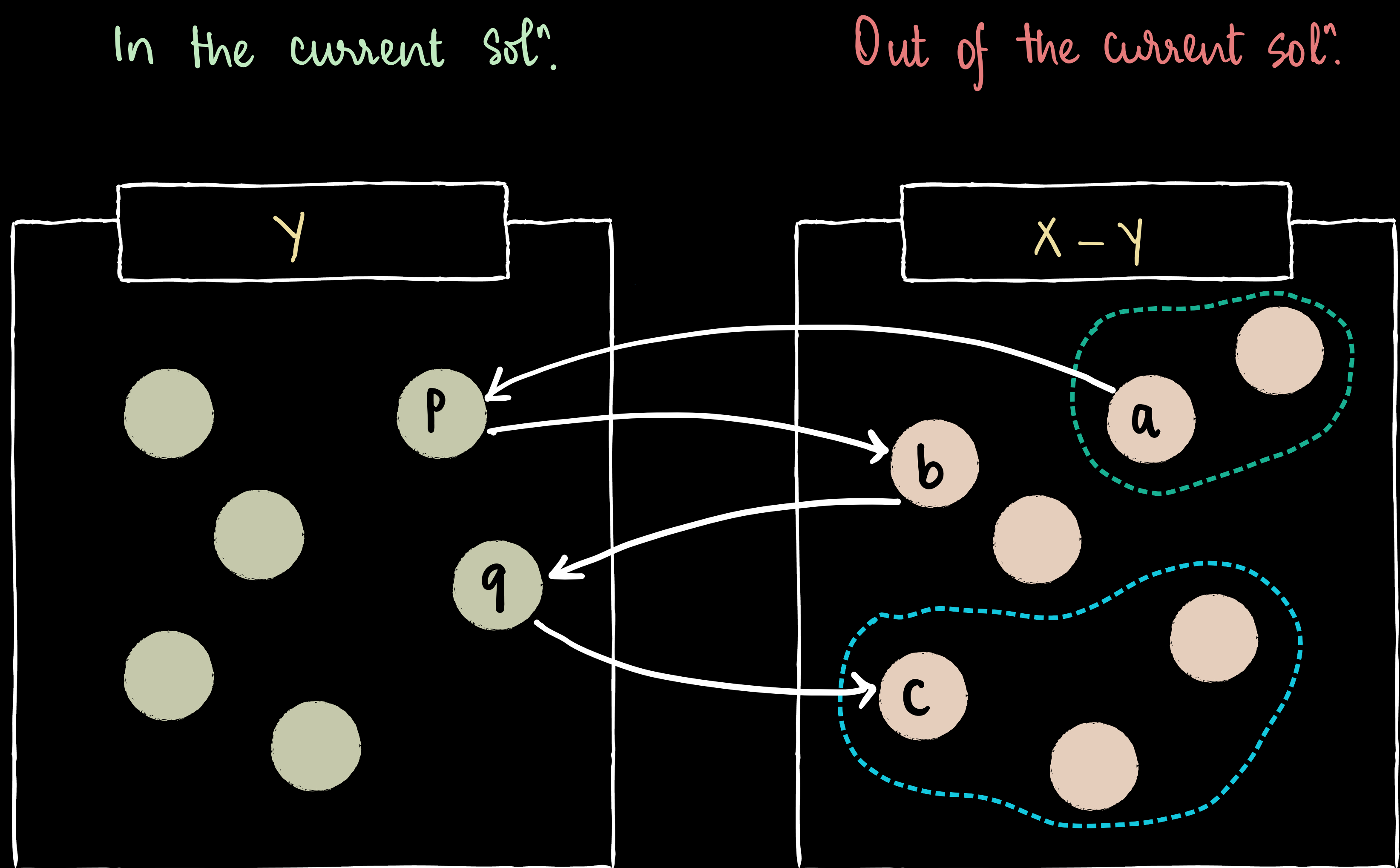
EA

$$Y \cup \{b\} \in I_1$$

$$Y \cup \{c\} - \{a\} \in I_1$$

$$Y \cup \{c\} \notin I_1$$

$$\Rightarrow Y \cup \{c\} - \{a\} \cup \{b\} \in I_1$$



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

A reasonable

Guess:

$$Y \cup \{a,b,c\} - \{p,q\} \in \mathcal{I}_1 \cap \mathcal{I}_2$$

More generally,

we have ...

$$\underbrace{\overset{\in Y_1}{b_0}\, a_1\, b_1\, a_2\, b_2\, a_3\, b_3 \cdots a_{k-1}\, b_{k-1}\, a_k\, \overset{\in Y_2}{b_k}}$$

<u>shortest $Y_1 - Y_2$ path.</u>

Claim. $Y - \{a_1, a_2, \dots, a_k\}$

$\cup \{b_0, b_1, \dots, b_k\} \in \mathcal{I}_1 \cap \mathcal{I}_2$

$\longrightarrow$ INDUCTION ON k

$$\underbrace{\overset{\in Y_1}{b_0}\ a_1\ b_1\ a_2\ b_2\ a_3\ b_3 \cdots a_{k-1}\ b_{k-1}\ a_k\ \overset{\in Y_2}{b_k}}$$

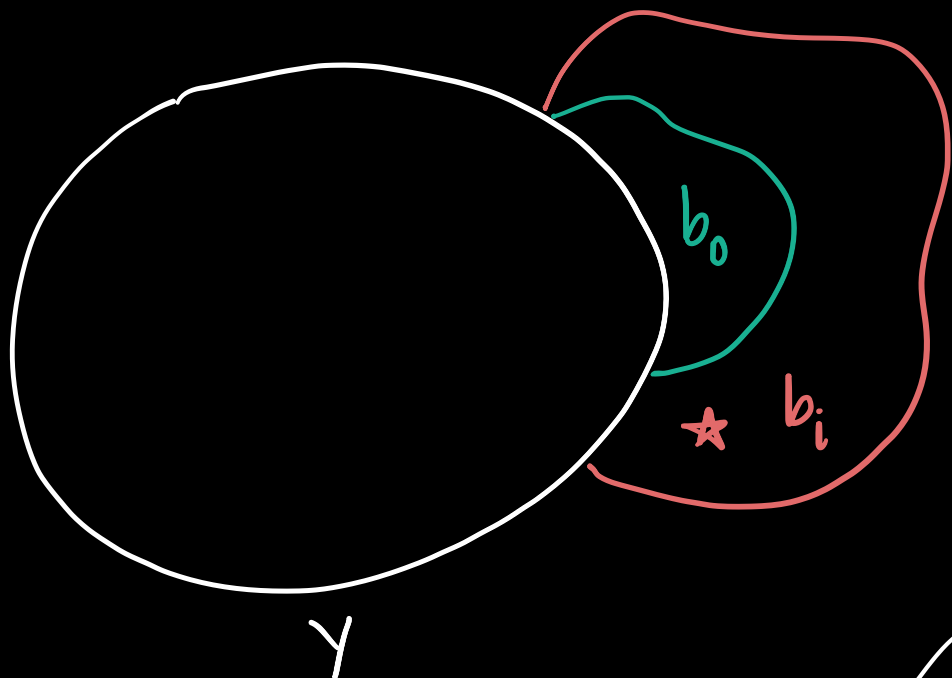
shortest $Y_1 - Y_2$ path.

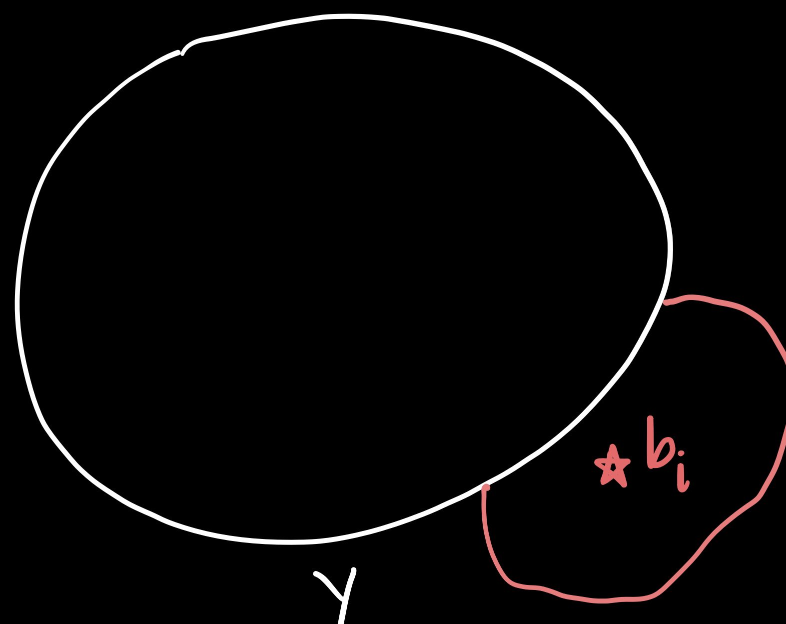
Observe:

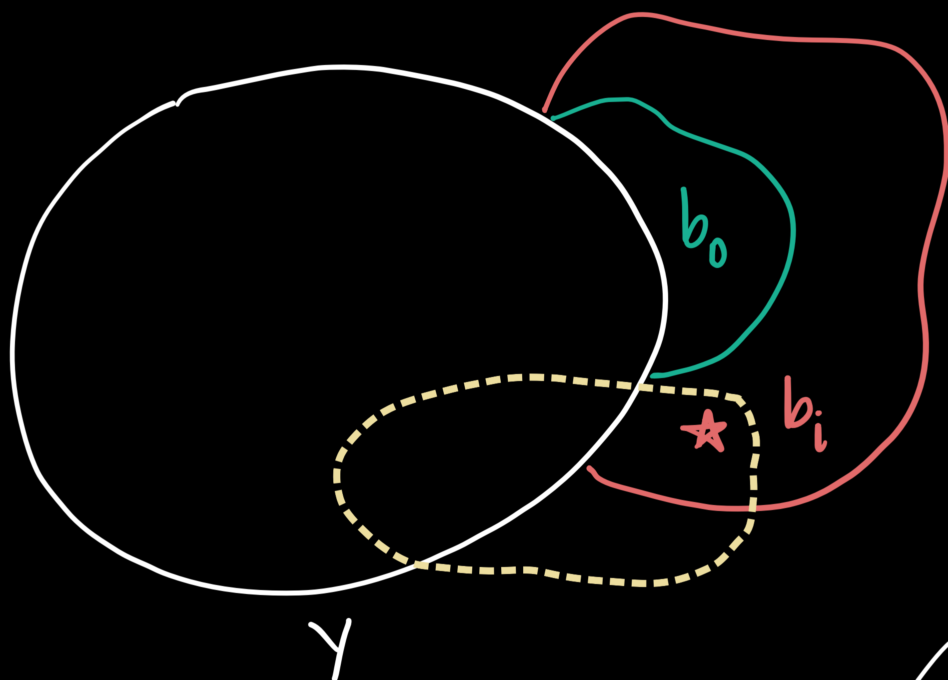
$$Y \cup \{b_0\} \in \mathcal{I}_1 \qquad \text{(by defn.)}$$

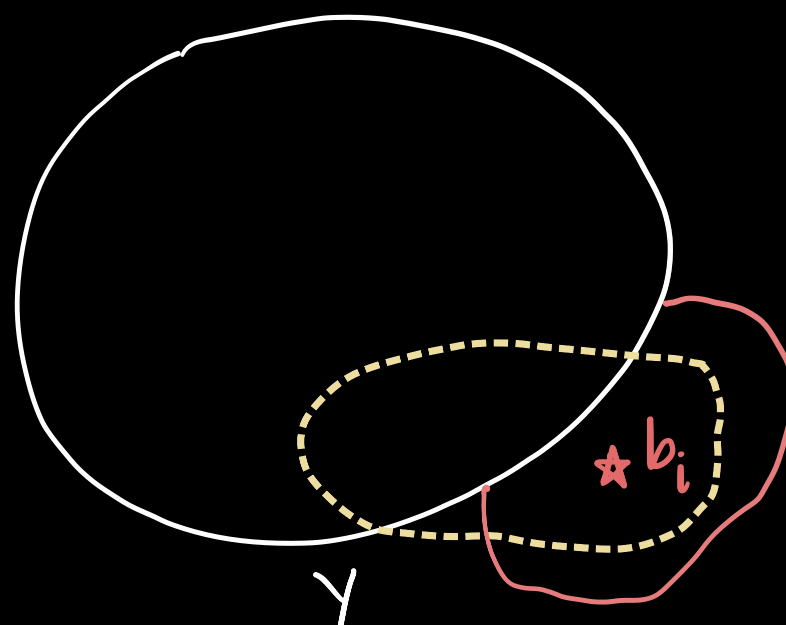
$$Y \cup \{b_0\} \cup \{b_i\} \notin \mathcal{I}_1 \quad \forall\, 1 \leq i \leq k \qquad \text{(shortest } Y_1 - Y_2 \text{ path)}$$

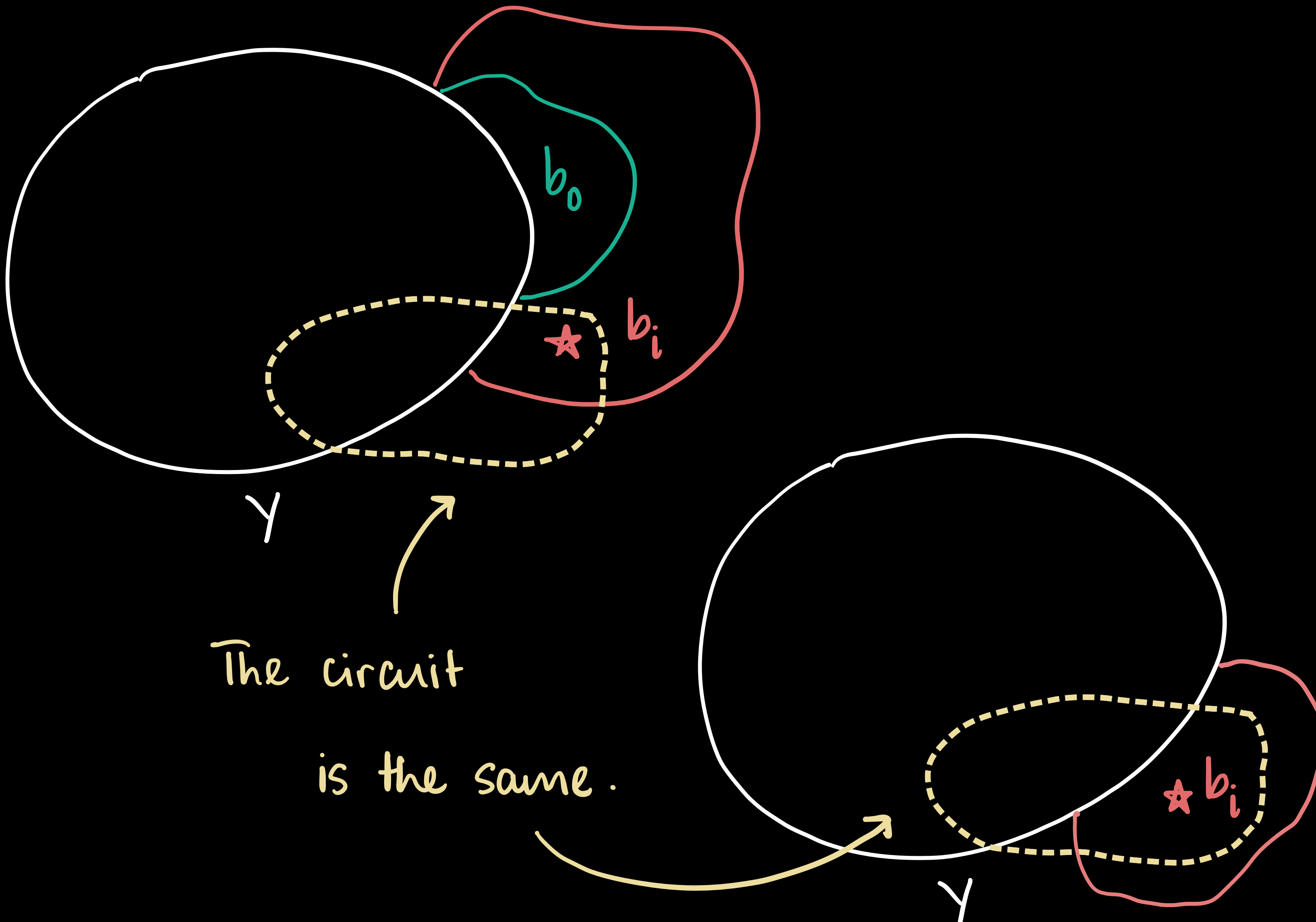
$$Y \cup \{b_i\} \notin \mathcal{I}_1 \quad \forall\, 1 \leq i \leq k \qquad \text{(shortest } Y_1 - Y_2 \text{ path)}$$


b_0
b_i
Y


b_i
Y

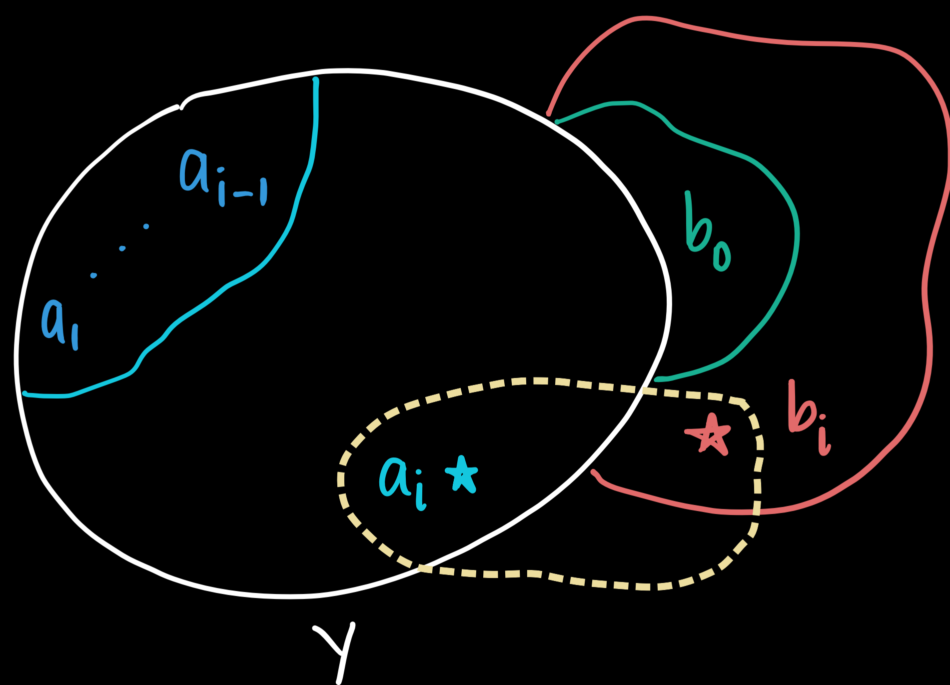

b_0
b_i
Y


b_i
Y

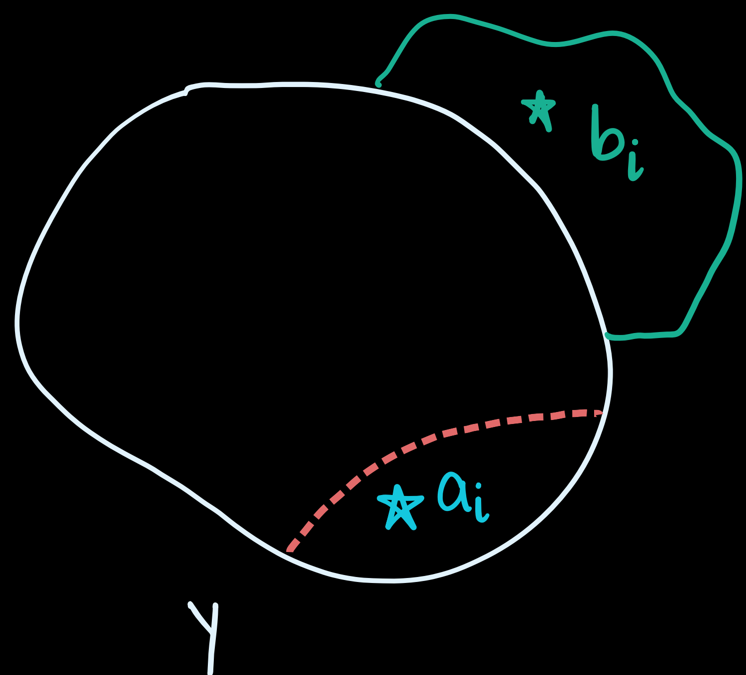


b0
bi
Y
The circuit
is the same.
bi
Y

Claim Let C be the unique circuit in $Y \cup \{b_0, b_i\}$.



$a_i \in C$ & $a_j \notin C$ $\forall$ $j \leq k$



$$\underbrace{Y - a_i + b_i \in \mathcal{I}_1}_{a_i b_i \in E}$$

$a_i \in$ any circuit of $Y \cup \{b_i\}$

$\Rightarrow$ $a_i \in$ any circuit of $Y \cup \{b_0, b_i\}$

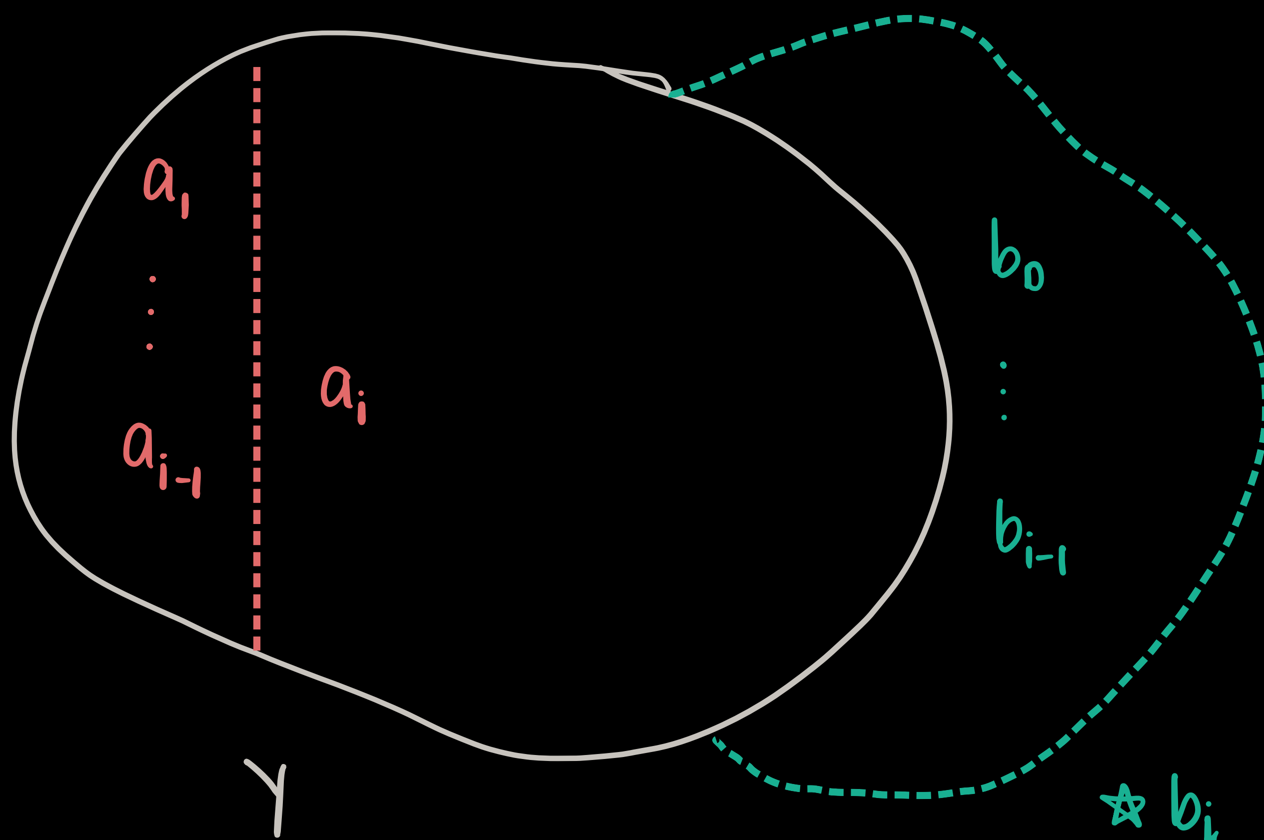
Induction Step

$$Y^{(i)} := Y - \{a_1, a_2, \dots, a_i\} \cup \{b_0, b_1, \dots, b_i\} \in \mathcal{I}_1 \cap \mathcal{I}_2$$

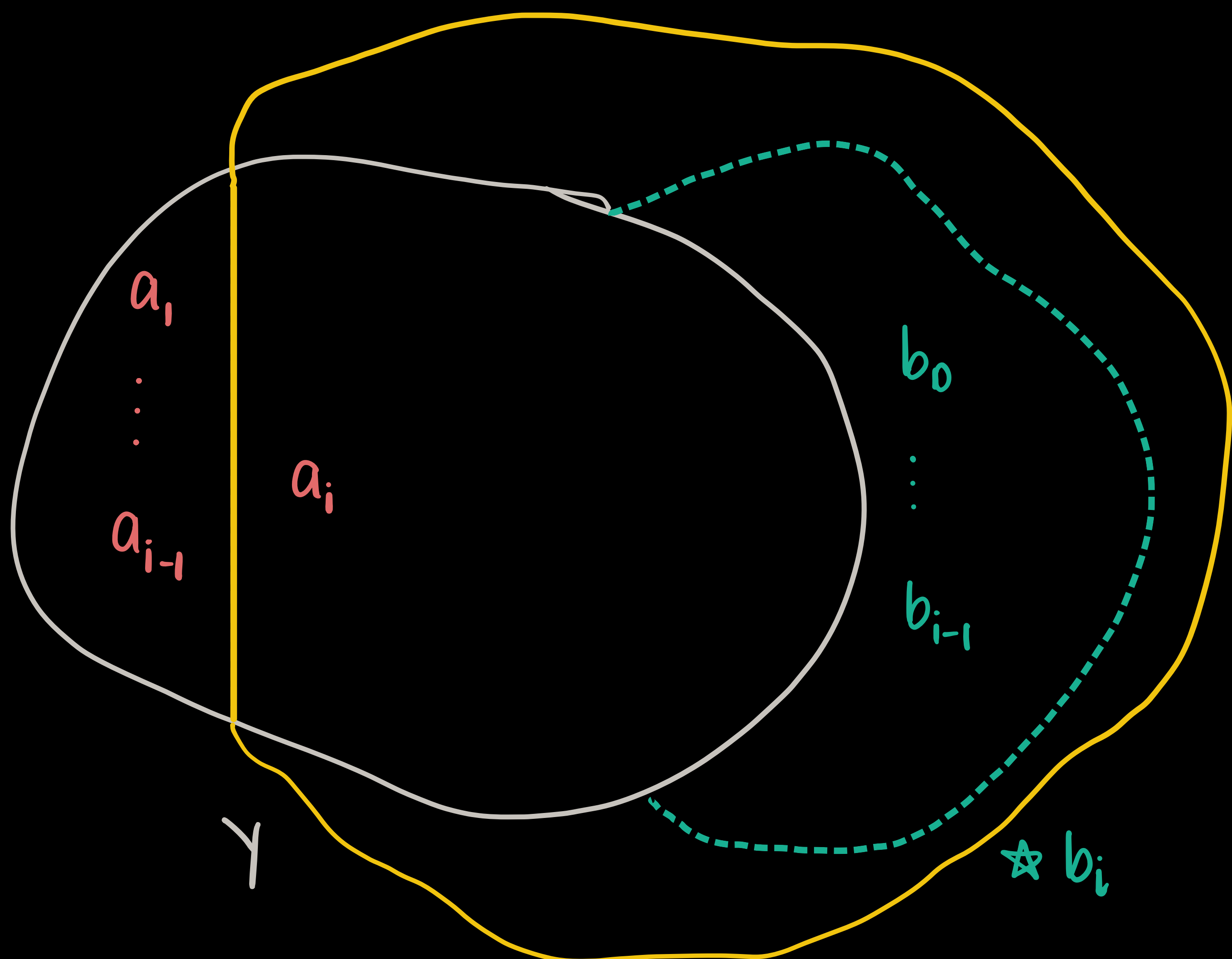
focus here.

I.H. $Y - \{a_1, a_2, \dots, a_{i-1}\} \cup \{b_0, b_1, \dots, b_{i-1}\} \in \mathcal{I}_1$

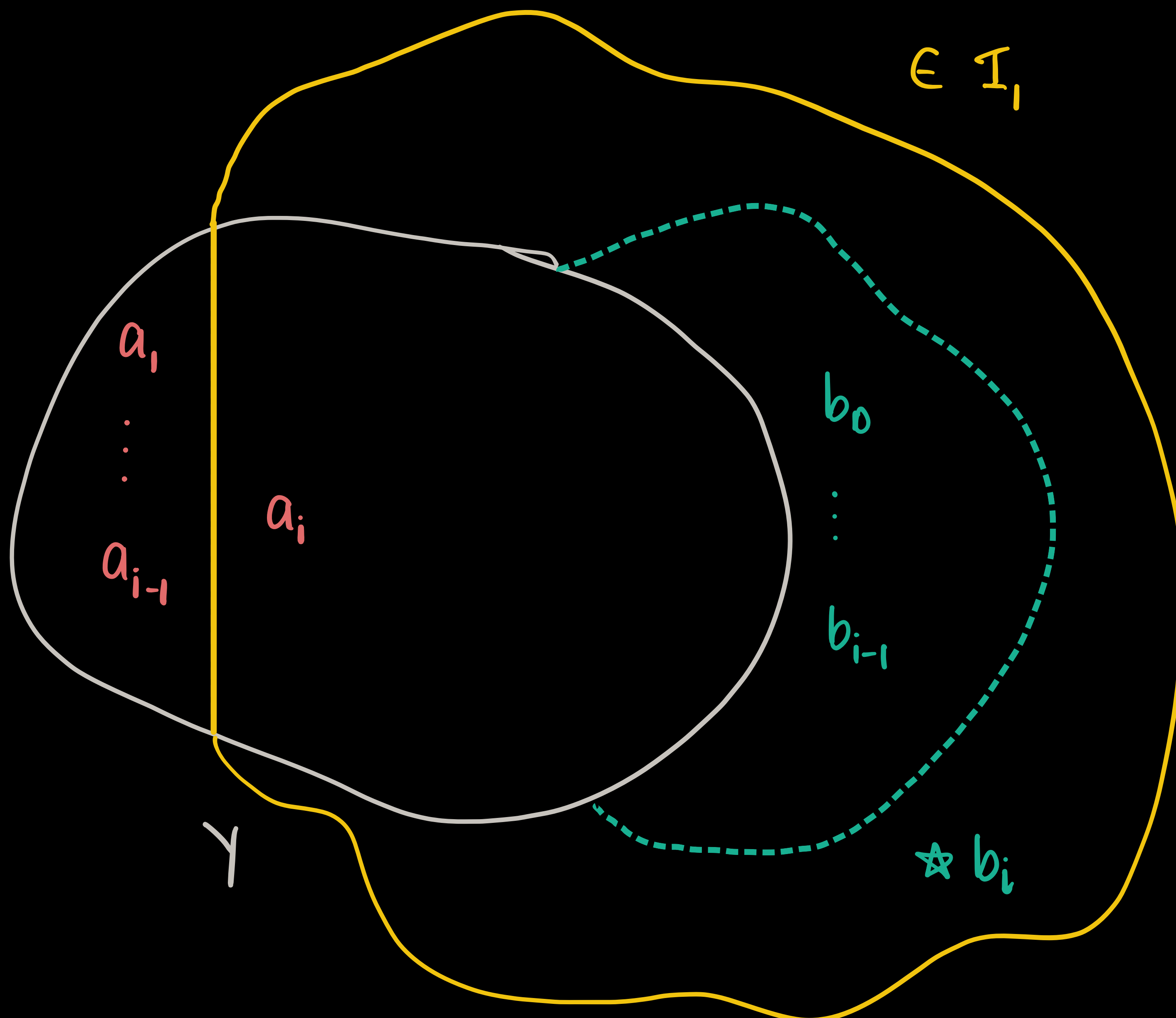
$Y^{(i-1)}$



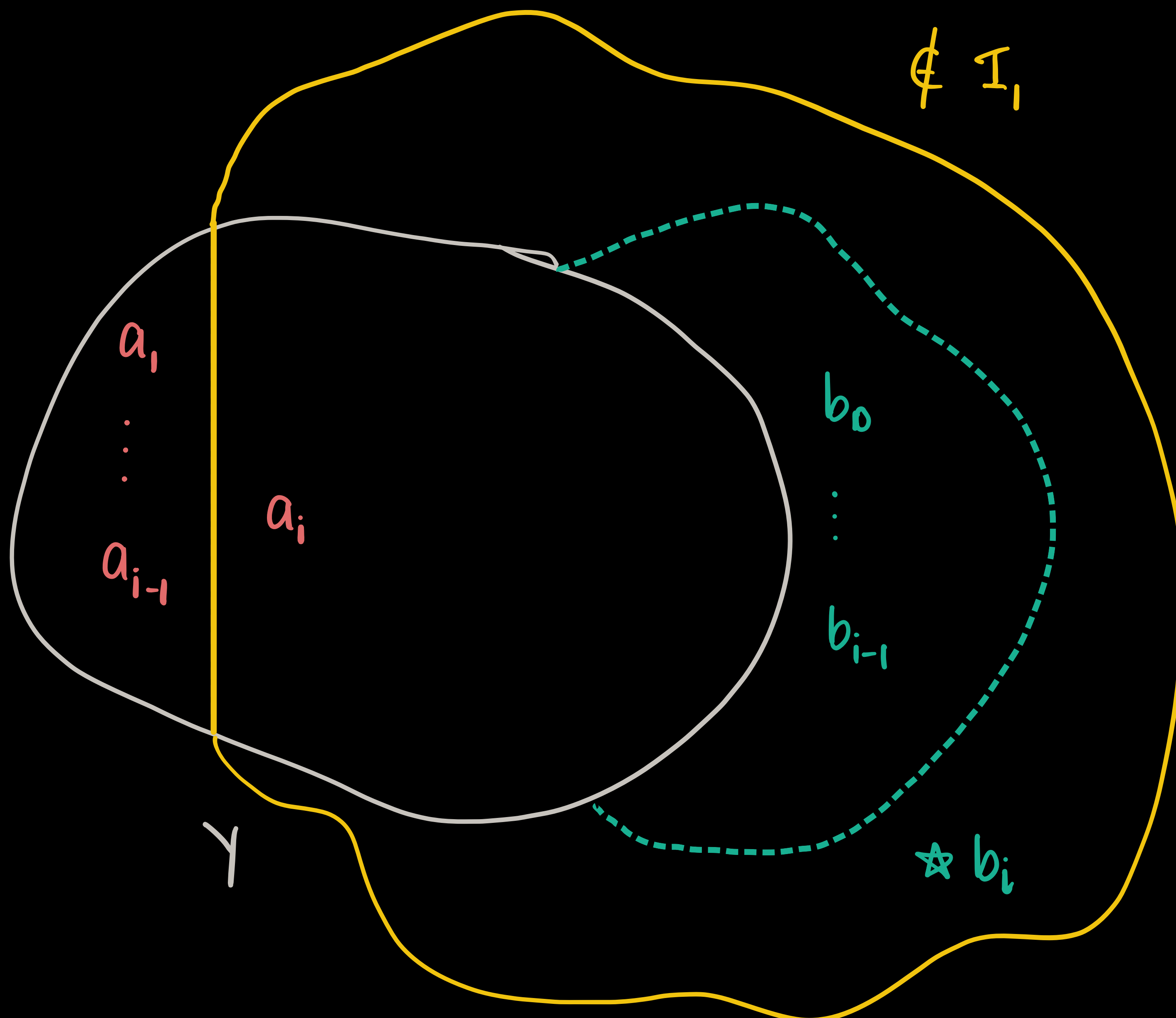
a_1
$\vdots$
a_{i-1}
a_i
b_0
$\vdots$
b_{i-1}
$\star\ b_i$
Y



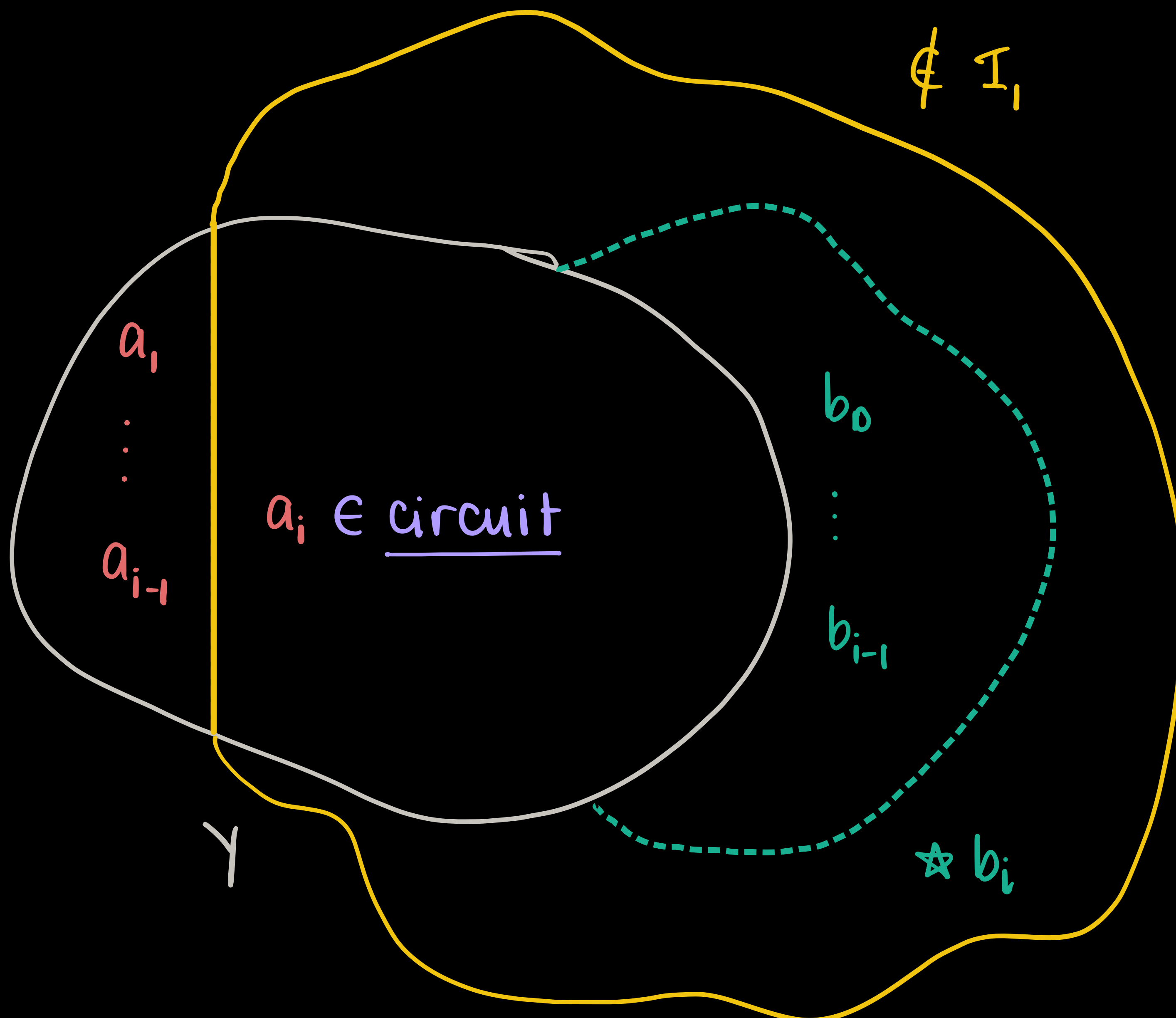
a_1
a_{i-1}
a_i
b_0
b_{i-1}
b_i
Y



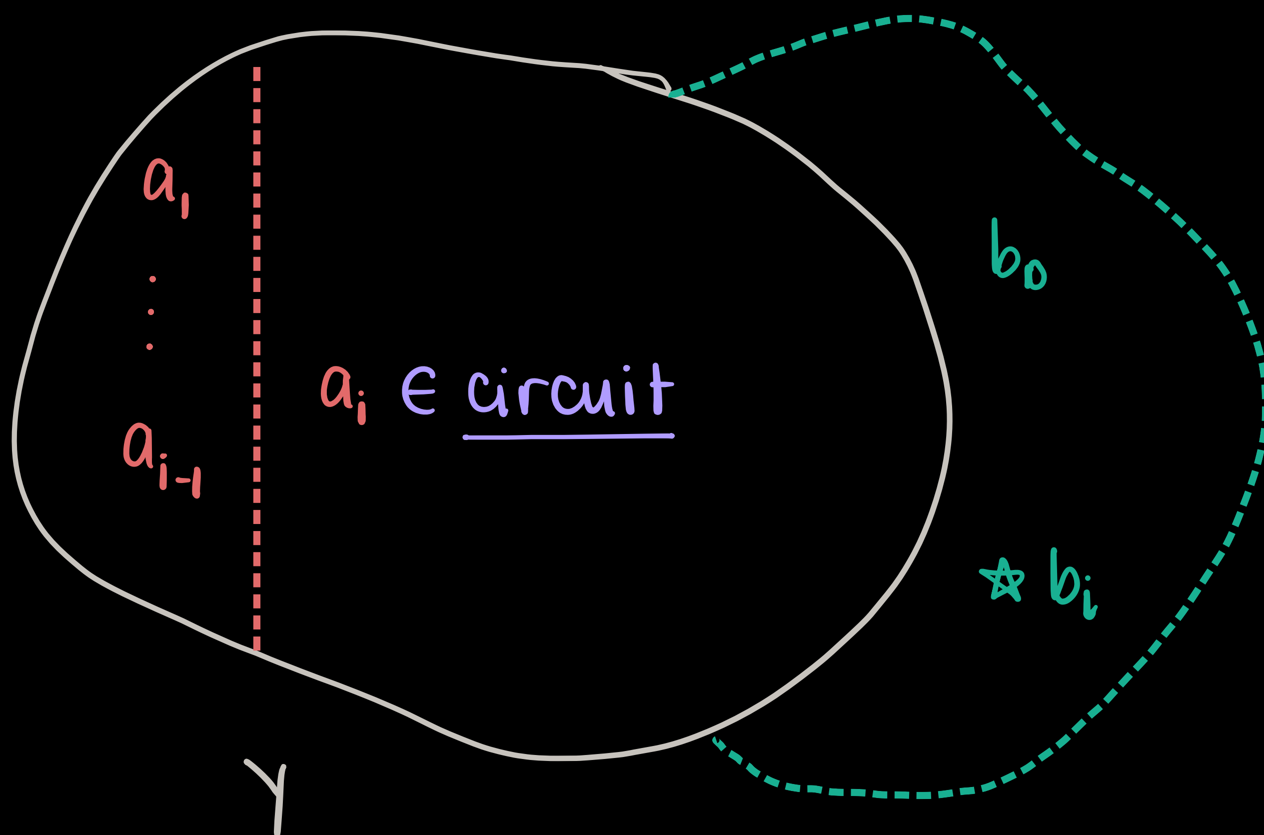
$\in \mathcal{I}_1$
a_1
$\vdots$
a_{i-1}
a_i
b_0
$\vdots$
b_{i-1}
$\star\ b_i$
Y



$\notin I_1$
a_1
$\vdots$
a_{i-1}
a_i
b_0
$\vdots$
b_{i-1}
$\star\, b_i$
Y



$\notin I_1$
a_1
$\vdots$
a_{i-1}
$a_i \in$ circuit
b_0
$\vdots$
b_{i-1}
$\star\ b_i$
Y



a_1
$\vdots$
a_{i-1}
$a_i \in$ circuit
b_0
b_i
Y