Lecture 7

Tuesday, December 6, 2022 14:21

Isolation lemma

Let U be a universe, $\mathcal{F}$ be a family of sets over U, $|U| = n$.

pick a random weight $w : U \to \{1 \ldots W\}$

Then

$$\Pr[\mathcal{F} \text{ has a unique min weight set}] \geqslant \frac{n}{W}$$

Proof: An element u is critical if

① u is in some minimum weight set

② if $w(u)$ is increased by 1 then u is no longer in any min weight set

Claim: $\mathcal{F}$ has a unique min weight set

$\downarrow \uparrow$

No element is critical

Pf of claim: Tutorial!

What is $\Pr[u \text{ is critical}]$?

assign random weights to all other elements first!

→ AT MOST ONE VALUE $\{1 \ldots W\}$ WILL MAKE u critical!

PF: suppose $w(u) = i$ makes u critical.

then $\forall i' < i$ $w(u) = i'$ does not make u critical because

________________ ??

$\forall i' > i$ $w(u) = i'$ does not make u critical because

??

✓

$\Pr[u \text{ is critical}] \leq \frac{1}{W}$!

$\Downarrow$ Union Bound over all $u \in U$

$\Pr[\exists \text{ critical vertex}] \leq \frac{U}{W}$!

■

K-path

In: G, k

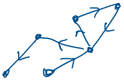
Task: Does G contain a simple k-path?

Saw earlier an algorithm with running time

$(2e)^k n^{O(1)}$

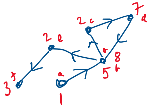
Now - $2^k n^{O(1)}$.

Walk: Sequence $v_1, v_2, v_3, \dots, v_\ell$ of vertices s.t. $v_i v_{i+1} \in E(G)$



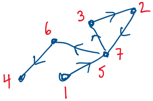
Labeled Walk

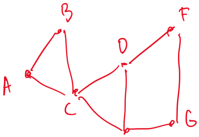
Walk $v_1 \dots v_\ell$ + $\sigma : \{1 \dots \ell\} \rightarrow \mathbb{N}$



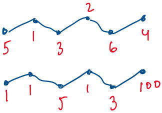
Multi-labeled Walk: σ is a bijection

$\{1 \dots \ell\} \rightarrow \{1 \dots \ell\}$





(Multi)-labeled PATH
(Multi)



We want to apply the isolation lemma to "the multi labeled k-paths of G"

⇝ Need a universe + family

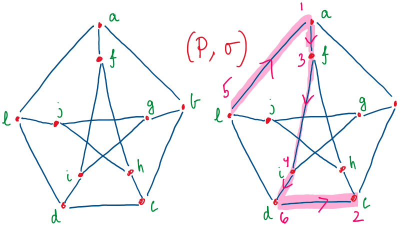
$U = E(G) \cup \{ (v,i) : v \in V(G), i \in \{1 \dots k\} \}$

??

Intuition: want to be able to assign weights to a path
- using an edge
- using a vertex with a given label

For a multi-labeled k-path $P = v_1 v_2 \dots v_k$, σ

Define $S(P,\sigma) = E(P) \cup \{ (v_i, \sigma(i)) : i \in \{1 \dots k\} \}$



$S(p,\sigma) = \{ ea, af, fi, id, dc, (e,5), (a,1), (f,3), (i,4), (d,6), (c,2) \}$

We defined U, now lets define $\mathcal{F}$

$\mathcal{F} = \{ S(P,\sigma) : (P,\sigma) \text{ is a multi-labeled k-Path in } G \}$

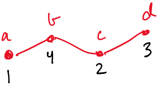
e

W, C, D, E, G, F

(W,σ)

$\sigma : \{1 \dots 6\} \to \{1 \dots 6\}$

$\sigma(1) = 4$
$\sigma(2) = 3$
$\sigma(3) = 1$
$\sigma(4) = 5$
$\sigma(5) = 2$
$\sigma(6) = 6$



(P,σ) = abcd, 1423
(Q,σ') = dcba, 3241

$S(P,\sigma)$ = ab, bc, cd, (a,1), (b,4), (c,2), (d,3)
$S(Q,\sigma')$ =

Pick a random weight function

$w: U \to 2\cdot|U|$

By isolation lemma

$$\Pr[\mathcal{F} \text{ has a unique min weight set}] \geq \frac{|U|}{2|U|} \geq \frac{1}{2}$$

For a labeled walk (Q,σ) with

$Q = q_1 \dots\dots q_\ell$

we define

$$\hat{w}(Q,\sigma) = \sum_{i=1}^{\ell-1} w(e) + \sum_{i=1}^{\ell} w(q_i,\sigma(i))$$

Note: for multi-labeled k-Paths (P,σ) we have

$$\hat{w}(P,\sigma) = w(S(P,\sigma))$$

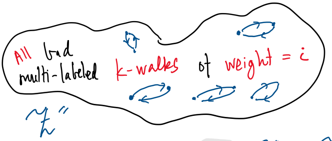
A multi-labeled k-walk (W,σ) is

good if W is a path

bad otherwise

LEMMA: $\forall i$, the number of bad multi labeled k-walks of weight i is even

Pf:



fixed point free involution $z: \mathcal{Z} \to \mathcal{Z}$

- Function from $\mathcal{Z}$ to itself
- fixed point free: $z(p) \neq p$
- involution: $z(z(p)) = p$

Take a bad walk $(Q,\sigma) \in \mathcal{Z}$

Let i,j be the lexicographically first pair so that

$q_i = q_j$

$$\begin{array}{lllllll} Q = & q_1 & q_2 \;\cdots & q_i & \cdots\; q_j \;\cdots & q(k) \\ \sigma = & \sigma(1) & \sigma(2)\;\cdots & \sigma(i) & \cdots\; \sigma(j) \;\cdots & \sigma(k) \\ & \shortparallel & \shortparallel & & & \shortparallel \\ \sigma' = & \sigma'(1) & \sigma'(2)\;\cdots & \sigma'(i) & \cdots\; \sigma'(j) \;\cdots & \sigma'(k) \end{array}$$

$$\begin{array}{rccccccc} - & a & b & c & d & b & c & a \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ - & 1 & 5 & 3 & 4 & 2 & 6 & 7 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

Set $\sigma' = \sigma$, but $\sigma'(i) := \sigma(j)$
$\sigma'(j) := \sigma(i)$

$$Z\,(Q, \sigma) := (Q, \sigma')$$

CHECKLIST

- Z maps bad multi-labeled k-walks
  to
  bad multi-labeled k-walks
- $\hat{w}(Q, \sigma) = \hat{w}(Z(Q,\sigma)) = \hat{w}(Q, \sigma')$
- Fixed point free: $\sigma' \neq \sigma$.
- Involution: $Z(Z(Q,\sigma)) = (Q, \sigma)$

Lemma: Let i be an integer, then
# multi-labeled k-PATHS of weight i is odd.
$\downarrow \uparrow$
# multi-labeled k-WALKS of weight i is odd

ALGORITHM

IDEA: Count \# multi-labeled walks of weight i
if ODD $\rightarrow$ $\exists$ multi-labeled k-path!

COUNTING MULTILABELED WALKS OF GIVEN WEIGHT.

$$T\,[v, S, i] =$$ \# labeled walks of weight i
ending in v
using each label in S once
and no other labels

$v \in V(G)$

$S \subseteq \{1 \ldots k\}$

$i \leq 2 \cdot |V| \cdot 2k$

$$T\,[v, S, i] = \begin{cases} 1 & \text{if } S = \{s\} \text{ and } w(v,s) = i \\ 0 & \text{if } S = \{s\} \text{ and } w(v,S) \neq i \\ \sum\limits_{\substack{s \in S \\ u \in N(v)}} T\big[u, S \setminus \{s\}, \underset{-\,w(uv)}{\underline{i - w(v,s)}}\big] & \end{cases}$$

Table size = $2^k \cdot O(knm)$

Time per entry: $O(k \cdot n)$

Total time: $2^k \cdot O(k^2 n^2 m)$

ALG for k-path outline:

```
• Pick random weight function
      $w: V \to \{1 \dots 2|V|\}$
• For $i = 1$ to $2|V| \cdot 2k$
      If # MULTI-LABELED K-WALKS OF
      WEIGHT $i$ is ODD
            RETURN "YES"
• RETURN "NO"
```

Steiner Tree:

In: G, $T \subseteq V(G)$

Task: Find Set $S \subseteq V(G)$ s.t

- $T \subseteq S$
- $G[S]$ is connected
- $|S|$ is minimized

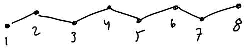
STRUCTURAL PARAMETERIZATION

Previously the parameter has always been OPT. Doesn't have to be!

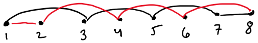
Parameter could also measure "complexity of the input instance"

Ex: BANDWIDTH $\leq k$

The vertices are ordered as $v_1 \dots v_n$ and $v_i v_j \in E(G) \to |i-j| \leq k$

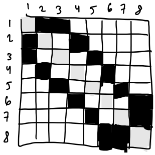


PATHS HAVE BW 1

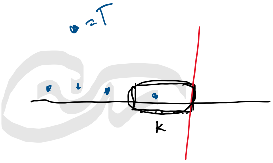


CYCLES HAVE BW 2

(we will assume the low bandwidth ordering is given as input)



All 1's in the adjacency matrix are in a "band"



Fairly simple Dynamic Programming algorithm for Steiner Tree on graphs of bandwidth k with running time $k^{O(k)} n^{O(1)}$

[TUTORIAL!]

CAN WE DO BETTER? LIKE $2^{O(k)} n^{O(1)}$

NEXT: YES!

$\boxed{\text{Using ISOLATION LEMMA}}$

Let s be the minimum size of a solution S

Set $U = V(G)$

$\mathcal{F} = \{ S : |S| = s \text{ and } S \text{ is a Steiner Tree for } T \}$

Pick random weight function $w: V(G) \to \{1 \dots 2n\}$

With probability $\geq 1/2$ $\mathcal{F}$ has a unique minimum weight set!

IDEA: FOR EVERY i, COUNT PAIRS S_1, S_2 OF DISJOINT VERTEX SETS SUCH THAT:

- $T \subseteq S_1 \cup S_2$
- NO EDGE FROM S_1 to S_2 in G
- WEIGHT OF $S_1 \cup S_2 = i$
- $|S_1 \cup S_2| = s$

OBSERVATION:

$$\#(S_1, S_2) = \sum_{S \subseteq V(G)} 2^{\#cc(G[S])}$$

$T \subseteq S$
$w(S) = i$

Thus, every STEINER TREE S of weight i contributes $2 \pmod 4$ while every disconnected S contributes $0 \pmod 4$!!!

ALG: FOR EVERY i, s COUNT PAIRS S_1, S_2 of DISJOINT VERTEX SETS SUCH THAT:

- $T \subseteq S_1 \cup S_2$
- No EDGE FROM S_1 to S_2 in G
- WEIGHT OF $S_1 \cup S_2 = i$
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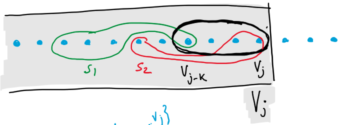

$$X[\underset{\le n}{\overset{\text{position}}{j}}, \underset{\le 2n^2}{\overset{\text{weight}}{i}}, \underset{\le n}{\overset{\text{size}}{s}}, \underset{\subseteq \{v_{j-k} \dots v_j\}}{\overset{S_1 \cap \{v_{j-k} \dots v_j\}}{S_1^*}}, S_2^*] = \# S_1, S_2 \subseteq V_j \text{ s.t}$$

- S_1, S_2 disjoint
- $T \cap V_j \subseteq S_1 \cup S_2$
- No edge from S_1 to S_2
- $w(S_1 \cup S_2) = i$
- $S_1 \cap \{v_{j-k} \dots v_j\} = S_1^*$
- $S_2 \cap \{v_{j-k} \dots v_j\} = S_2^*$

SANITY CHECK FOR S_1^*, S_2^*:

- $S_1^* \cap S_2^* = \emptyset$
- $T \cap \{v_{j-k} \dots v_j\} = S_1^* \cup S_2^*$
- No edge from S_1^* to S_2^*

S_1^*, S_2^* are "good for position j"

RECURRENCE:

$$X[\overset{\text{position}}{j}, \overset{\text{weight}}{i}, \overset{\text{size}}{s}, \overset{S_1 \cap \{v_{j-k} \dots v_j\}}{S_1^*}, S_2^*] =$$

remove v_j	add v_{j-k-1} to S_1^*	add v_{j-k-1} to S_2^*	
Y	Y	Y	NA
Y	Y	N	a
Y	N	Y	b
Y	N	N	c
N	Y	Y	NA
N	Y	N	d
N	N	Y	e
N	N	N	f

$(a) + (b) + (c) + (d) + (e) + (f)$

ⓐ $X[j-1, i-w(v_j), s-1, S_1^* \cup \{v_{j-k+1}\}, S_2^*]$

$$\cdot \begin{cases} 1 & \text{if } S_1^* \cup \{v_{j-k+1}\}, S_2^* \text{ good for } j-1 \\ 0 & \text{otherwise} \end{cases}$$

Size of table: $3^k \cdot \text{poly}(n)$

Time per entry: $\text{poly}(n)$

$\Longrightarrow 3^k n^{O(1)}$ time!

Overall Algorithm outline:

```
Pick random weight function
    $w : V(G) \to \{1 \dots 2n\}$
For every $s$ from 1 to $n$
    For every $i \in \{1 \dots 2n^2\}$
        Compute # pairs $(S_1, S_2)$ s.t
            • $S_1 \cap S_2 = \emptyset$
            • $T \subseteq S_1 \cup S_2$
            • No EDGE FROM $S_1$ to $S_2$ in $G$
            • WEIGHT OF $S_1 \cup S_2 = i$
        if $\#(S_1, S_2) \equiv 2 \mod 4$
            RETURN $s$
```

- Runtime $3^k n^{O(1)}$
- Success prob ½ :
  - if alg returns $s \to \exists$ steiner tree of size $\leq s$
  - if $\exists$ steiner tree of size $t \to$ with prob $\geq$ ½ $s \leq t$