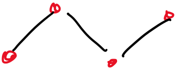
Lecture 4

Sunday, December 4, 2022 21:04

In: graph G, integer k

Q: Does G have a path on k vertices??



Easy: $n^k \cdot k$

Is it FPT - $f(k)\,n^{O(1)}$??

↓

Yes - will see $(2e)^k n^{O(1)}$!

STEP 1 RANDOMLY COLOR $V(G)$ with k colors $\{1.....k\}$

(pick $c: V(G) \to \{1...k\}$ u.a.r.)

Good event:

At least one k-path P (if it exists) gets MULTICOLORED

All vertices of P got different colors

$$\Pr[P \text{ gets multicolored}] \geq \frac{k!}{k^k} \geq \frac{1}{e^k}$$

Claim: $\frac{k^k}{k!} \le e^k$

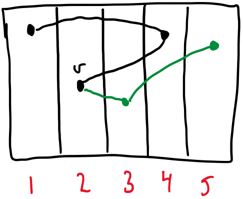
Pf: $\frac{1^1}{1!} \le e^1$

$$\frac{k^k}{k!} = \frac{(k-1)^{k-1}}{(k-1)!} \cdot \frac{(k-1)!}{(k-1)^{k-1}} \cdot \frac{k^k}{k!}$$

$$\le e^k \cdot \frac{1}{k} \cdot \frac{k^{k-1}}{(k-1)^{k-1}} \cdot k$$

$$= e^{k-1} \cdot \left(1 + \frac{1}{k-1}\right)^{k-1} \le e^{k-1} \cdot e = e^k$$

Alg for <u>finding</u> a multi-colored path?



$$T[v, S] = \begin{cases} \text{TRUE} & \text{if } \exists \text{ path ending in } v \text{ using each color in } S \text{ \underline{precisely} once} \\ \text{FALSE} & \text{otherwise} \end{cases}$$

$\uparrow$

$S \subseteq \{1 \ldots k\}$

$$T[v,S] = \begin{cases} \text{TRUE} & \text{if } S = \{col(v)\} \\ \text{FALSE} & \text{if } col(v) \notin S \\ \bigvee_{u \in N(v)} T[u, S - \{col(v)\}] & \text{otherwise} \end{cases}$$

Table has size $n \cdot 2^k$

Time $O(n)$ to compute $T[v,S]$

Total time $O(2^k n^2)$!

COUNTING k-paths: Pick random coloring

Count multi-colored k-paths

time: $2^k n^2$

success prob: $1/e^k$ $\longrightarrow$ $(2e)^k n^2$ time alg!

Ok-ok so we can DETECT k-paths in graphs... but can we COUNT them?

the alg for DETECTING multicolored paths generalizes to COUNTING them!

$T[v, S]$ = # paths ending in v using each color in S precisely once

$\uparrow$

$S \subseteq \{1 \ldots k\}$

$$T[v,S] = \begin{cases} 1 & \text{if } S = \{col(v)\} \\ 0 & \text{if } col(v) \notin S \\ \sum_{u \in N(v)} T[u, S - \{col(v)\}] & \text{otherwise} \end{cases}$$

Ok ... so some hope for counting k-paths? (not multi colored)

NO ☹ → Counting k-paths exactly in time $f(k)\, n^c$ would imply $\#W[1] = FPT$.

Hmm... ok can we maybe $(1 \pm \varepsilon)$-APPROXIMATELY count them in time $1/\text{poly}(\varepsilon) \cdot f(k)\, n^c$?

↳ HARDER THAN FINDING ONE!

Digression: CHERNOFF BOUNDS

if X is the sum of many independent random variables with "small" values

$\Rightarrow$ X is very likely to be **very** close to $E[X]$

Thm: Let $X = X_1 + X_2 \dots + X_n$ where:

- X_i's take values from $\{0,1\}$
- X_i's are independent

then $$\Pr\left[\left|X - E[X]\right| \geq \varepsilon E[X]\right] \leq 2e^{-\varepsilon^2 \cdot E[X]/3}$$

A useful variant:

Thm: Let $X = X_1 + X_2 \dots + X_n$ where:

- X_i's take values from $\{0,1\}$
- X_i's are independent
- Let $\mu_L \leq E[X] \leq \mu_H$

Then:

$$\Pr\left[X - \mu_H \geq \varepsilon \mu_H\right] \leq e^{-\varepsilon^2 \cdot \mu_H/3}$$

$$\Pr\left[\mu_L - X \geq \varepsilon \mu_L\right] \leq e^{-\varepsilon^2 \cdot \mu_L/3}$$

Application to APPROXIMATELY COUNTING # k-paths.

```
ALG $(G, k, \varepsilon)$
    $a = 0$
    For $i = 1$ to $t$          ← Choose $t$ based on $k, n, \varepsilon$
        pick random $k$-coloring $col_i$ of $G$
        $a = a + \#$ multi-colored $k$-paths
    RETURN $a \cdot \frac{k^k}{k!} \cdot \frac{1}{t}$
```

$$\boxed{t = \frac{k^k}{k!} \cdot \frac{k \cdot \log n \cdot 6}{\varepsilon^2}}$$

RUNTIME: $t \cdot 2^k \cdot n^2$

$$= O\left((2e)^k \, n^2 \log n / \varepsilon^2\right)$$

Let $\#KP$ be the number of k-paths in G.

Let $X_{P,i}$ be a random variable that is 1 if the path P is multi-colored in round i

$$a = \sum_{i=1}^{t} \sum_{P} X_{P,i}$$

and set

$$X_P = \sum_{i=1}^{t} X_{P,i} = \# \text{rounds where } P \text{ is multi-colored}$$

So $$E[a] = \sum_{i=1}^{t} \sum_{P} E[X_{P,i}]$$

$$= \sum_{i=1}^{t} \sum_{P} \frac{k!}{k^k} = t \cdot \#KP \cdot \frac{k!}{k^k}$$

Note that if a is close to $E[a]$ then ALG output is close to $\#KP$!

IDEA: WANT TO APPLY CHERNOFF BOUNDS TO a (Since it is sum of $X_{i,p}$-s)

WHY DOES THIS NOT WORK?

. . .

Instead, apply Chernoff to X_p for each P!

For every P, $E[X_p] = \sum_{i=1}^{t} E[X_{p,i}] = t \cdot \frac{k!}{k^k}$

X_p is a sum of ind. 0-1 vars.

Chernoff:

$$\Pr\left[\,|X_p - E[X_p]| \geq \varepsilon E[X_p]\right] \leq 2e^{-\varepsilon^2 E[X_p]/3}$$

$$= 2e^{-\varepsilon^2 t \cdot \frac{k!}{k^k}/3}$$

$$= 2e^{-\frac{\varepsilon^2}{3} \cdot \frac{k!}{k^k} \cdot \frac{k^k}{k!} \cdot \frac{6}{\varepsilon^2} \cdot k \cdot \log n}$$

$$= 2e^{-2k \log n}$$

$$= 2/n^{2k}$$

By union bound over all $\le \binom{n}{k}$ choices for P

$$\Pr\left[\exists P : |X_P - E[X_P]| \geq \varepsilon E[X_P]\right] \leq \binom{n}{k} \cdot 2/n^{2k}$$

$$\leq 1/2$$

Suppose bad event above does not happen

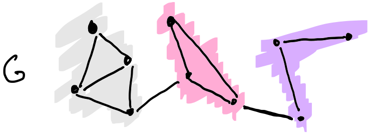
then $a = \sum_P X_P \rightarrow |a - E[a]| \leq \varepsilon E[a]$

Chernoff + union = killer combination

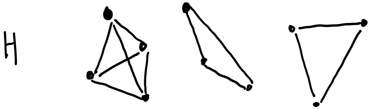
d-Clustering

In: Graph G, integer k

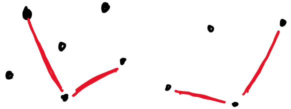
Q: $\exists$ d-cluster graph H s.t $|G \triangle H| \leq k$?



d = 3



$G \triangle H$



NP-COMPLETE (ALREADY FOR $d=2$!)

Will see a $d^{O(\sqrt{k})}$ time algorithm!

Coloring Lemma:

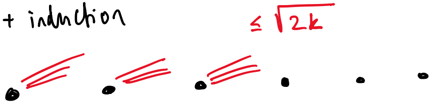
Let G be a graph with k edges, and let $q = \sqrt{2k} \cdot 2$ if we color G u.a.r with q colors then $\Pr[G \text{ is properly colored}] \geq$

Pf: Claim 1: G has a vertex of degree $\leq \sqrt{2k}$

pf: otherwise $2k$ = sum of degrees is $> \sqrt{2k} \cdot \sqrt{2k} = 2k$ ↯

Claim 2: G is $\sqrt{2k}$-degenerate

pf: 1 + induction



Proof of lemma:

Let $d_1, d_2 \dots d_n$ be the forward degrees of vertices in the $\sqrt{2k}$ degeneracy sequence

$$\Pr\left[\text{col is proper } q \text{ coloring}\right] \geqslant \prod_{i=1}^{n}\left(1-\frac{d_i}{q}\right) \geqslant \prod_{i=1}^{n} 4^{-d_i/q}$$

$$= 4^{-\Sigma d_i/q} = 4^{-k/q} = 4^{-k/2\sqrt{2}\sqrt{k}}$$

$$= 4^{-\sqrt{k}/2\sqrt{2}} = 1/2^{\sqrt{k/2}}$$

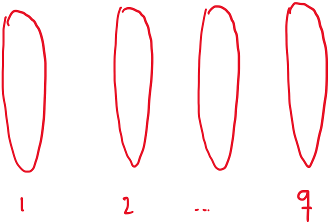
Algorithm for d-clustering

1: $q = 2\cdot 2\sqrt{k}$

2: RANDOMLY COLOR $V(G)$ WITH q COLORS

[WITH PROB $\geqslant 1/2^{\sqrt{k/2}}$

$S = G \triangle H$ is properly colored !]



EACH COLOR CLASS IS ALREADY A UNION OF $\leq d$ CLIQUES [WHY ?]

For each clique in each color class, guess which cluster it belongs to

Time: $d^{dq} = d^{d\sqrt{2k}}$

Success prob: $1/2^{\sqrt{k/2}}$

Total time:

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probably stop here

In: Tournament T, k

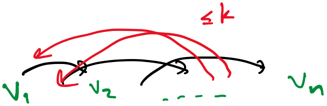
TASK: Find set $S \subseteq E(T)$ s.t

$T-S$ is acyclic & $|S| \leq k$

Equivalent: rank vertices of T as

$v_1, v_2, \dots v_n$

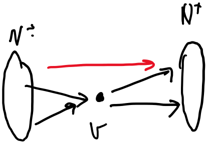
s.t $\leq k$ edges goes from "right to left"



EQUIVALENT
- REVERSE
S

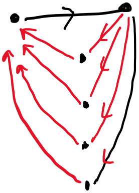
NP-complete!

What about solution of size $\leq k$?



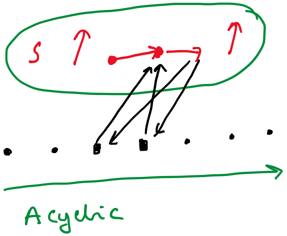
STEP 1: Reduce n to $O(k^3)$:

- if no edges from $N^+(v)$ to $N^-(v)$ → remove v
- If e is in $\geq k+1$ triangles, reverse v and decrease k by 1



IF ∃ SOLN S OF SIZE $\leq k$ AND NO R.R. APPLIES

↪ If $V(G) \geq 2k + k^2$



ENDPOINTS OF S $\leq 2k$

$> k^2$ vertices → some S-edge is in $> k$ triangles