Lecture 2 - Linear Programming
Friday, December 2, 2022 04:25

Integer Linear Program:

n variables $x_1, x_2 \dots x_n \in \mathbb{Z}$

a linear objective function to be optimized

$$\text{Max/Min} \sum_{i=1}^{n} c_i \cdot x_i$$

$c_1 \dots c_n$ some constants $\in \mathbb{Z}$ or $\mathbb{Q}$

subject to m linear inequalities:

$$a_1^1 x_1 + a_2^1 x_2 + a_3^1 x_3 \dots + a_n^1 x_n \le b_1$$
$$a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_3 \dots + a_n^2 x_n \ge b_2$$
$$a_1^3 x_1 + a_2^3 x_2 + a_3^3 x_3 \dots + a_n^3 x_n = b_3$$

a_i^j and b_i some constants $\in \mathbb{Z}$ or $\mathbb{Q}$

* FEASIBLE SOLUTION ← SATISFIES ALL CONSTRAINTS
* VALUE OF SOLUTION ← DUH
* OPTIMAL SOLUTION TO ILP / INSTANCE (MAY NOT EXIST)
  - FEASIBLE
  - NO SOLN IS BETTER
* OPTIMAL VALUE

ILP: In: • Min/Max, n, m
• $c_1 \dots c_n$
• $a_1^1, a_2^1, \dots a_1^2, a_2^2, \dots a_n^m$
• $b_1 \dots b_m$

TASK: COMPUTE OPTIMAL SOLN
(OR DETERMINE INFEASIBLE OR UNBOUNDED)

Note: if M is max numerator / denominator then input size is $O(n \cdot m \cdot \log M)$

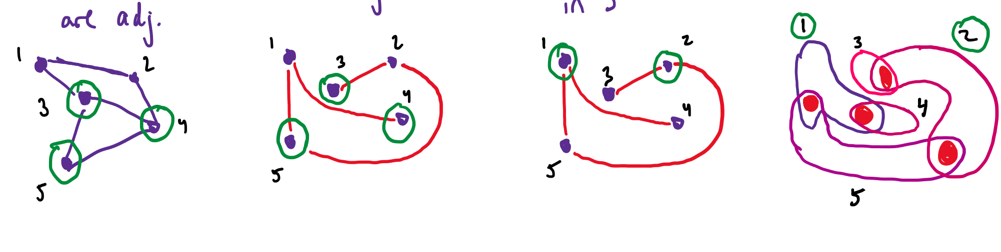
SOME OBSERVATIONS:
- Wlog [WRT POLY TIME] all inputs in $\mathbb{Z}$ (why?)
- Wlog all constraints are $\le$ or $\ge$
- ... actually just $\le$. (why?)

PRACTICAL INFO:
ILP SOLVERS SOLVE (MANY) ILP's RIDICULOUSLY FAST

THEORETICAL INFO:
ILP is NP-COMPLETE

CLIQUE → ISET → V.C. → SET COVER

CLIQUE: G, k — clique of size $\ge k$? — every two vertices in S are adj.
ISET: G, k — ind. set of size $\ge k$? — no two vertices in S are adj.
V.C.: G, k — vertex cover of size $\le k$? — Every edge has ≥ 1 endpoint in S
SET COVER: $U, \mathcal{F}, k$ — set cover of size $\le k$? — Every element appears in ≥ 1 set



VERTEX COVER → ILP reduction

In: G
Task: find $S \subseteq V(G)$ s.t $|S|$ is minimized
$\forall uv \in E(G): \{u,v\} \cap S \neq \emptyset$

→ $\forall$ vertex v of G, variable x_v
$0 \le x_v \le 1$, $x_v \in \mathbb{Z}$ ($v \in S$)
$$\min \sum_{v \in V(G)} x_v \quad \text{s.t}$$
$$\forall uv \in E(G): x_u + x_v \ge 1$$

Vertex Cover $\xrightarrow{\text{SIZE} \quad \text{VALUE}}$ Feasible Solution
Feasible Solution $\xrightarrow{\text{VALUE} \quad \text{SIZE}}$ Vertex Cover

$\boxed{\text{Linear Programming}}$ Same as ILP, but now variables are in $\mathbb{R}$ instead of $\mathbb{Z}$

MIRACLE THM:
LP in polynomial time!
Khachiyan '79: $n^7 m \cdot \log(M)$
Karmarkar '84: $n^{4.5} m \cdot \log(M)$
... (FASTER ALGS EXIST) ...

n = # vars
m = # constraints
M = largest integer in input.

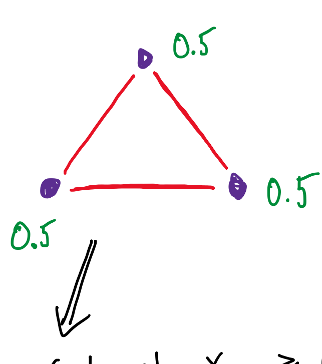
For every ILP there is a corresponding LP RELAXATION
SAME VARS, CONSTRAINTS & OBJECTIVE FUNC., BUT VARS $\in \mathbb{R}$ instead of $\mathbb{Z}$

Note: EVERY FEASIBLE SOLUTION TO AN ILP IS A FEASIBLE SOLUTION TO THE LP-RELAXATION

FOR MINIMIZATION $OPT_{ILP} \ge \le OPT_{LP}$?
MAXIMIZATION $OPT_{ILP} \ge \le OPT_{LP}$

!! WHAT IF WE SOLVE THE LP RELAXATION OF THE VERTEX COVER ILP ??

$\forall$ vertex v of G, variable x_v
$0 \le x_v \le 1$, $x_v \in \mathbb{R}$
$$\min \sum_{v \in V(G)} x_v \quad \text{s.t}$$
$$\forall uv \in E(G): x_u + x_v \ge 1$$



what to do??

$$x_v^* = \begin{cases} 1 & \text{if } x_v \ge 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Vertex Cover
$S = \{v : x_v^* = 1\}$
$= \{v : x_v \ge 0.5\}$

① Is S A VERTEX COVER?
② How DOES $|S|$ compare to the size of a minimum vertex cover OPT for G?

① S is a vertex cover because $\forall uv \in E(G)$ $x_u \ge 0.5$ or $x_v \ge 0.5$
② How good is S?
$$|S| = \sum_{v \in V(G)} x_v^* \le \sum_{v \in V(G)} x_v \cdot 2 = 2 \cdot OPT_{LP} \le 2 \cdot OPT_{ILP} = 2 \cdot OPT \;!$$

S is never more than a factor 2 worse than optimal solution!

YOUR PROBLEM → ILP → SOLVE LP RELAXATION → ROUND TO INTEGRAL SOLUTION → FEASIBLE SOLUTION TO INSTANCE

Umm: Isn't this a RANDOMIZED algorithms course??

Set Cover
In: U ($|U| = n$)
$F = \{S_1, S_2 \dots S_m\}$ ($S_i \subseteq U$)
TASK: FIND $\mathcal{F}' \subseteq \mathcal{F}$ st
$C(\mathcal{F}') := \bigcup_{S \in \mathcal{F}'} S = U$
AND $|\mathcal{F}'|$ is minimized

ILP Formulation
VAR x_S for every $S \in \mathcal{F}$
$x_S = 1 \Rightarrow S \in \mathcal{F}'$

(*) $$\min \sum_{S \in \mathcal{F}} x_S$$
$$\text{s.t } \forall u \in U \sum_{S \in \mathcal{F} \text{ s.t } u \in S} x_S \ge 1$$
$$x_S \ge 0$$

Solve LP relaxation of (*) in poly time!

How to round?!!
For every S we have $0 \le x_S \le 1$ why?

Add S to $\mathcal{F}'$ with probability x_S ??!!

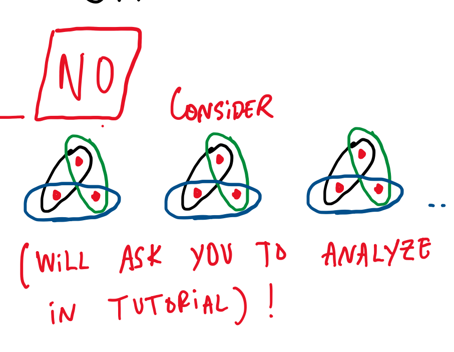
→ How big is $\mathcal{F}'$? → WHAT IS $E[\mathcal{F}']$
→ What is $Pr[\mathcal{F}' \text{ covers } U]$

How big is $\mathcal{F}'$? → WHAT IS $E[\mathcal{F}']$

Method of indicators:
let $Y_S = \begin{cases} 1 & \text{if } S \in \mathcal{F}' \\ 0 & \text{otherwise} \end{cases}$ $\boxed{E[Y_S] = x_S}$

$$E[|F'|] = E\Big[\sum_{S \in \mathcal{F}} Y_S\Big] = \sum_{S \in F} E[Y_S] = \sum_{S \in F} x_S = OPT_{LP} \le OPT$$

Does F' cover U? ← $\boxed{\text{NO}}$ CONSIDER



(WILL ASK YOU TO ANALYZE IN TUTORIAL)!

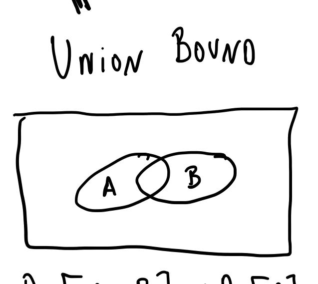
But can we say anything about what F' covers? YES!
Let $u \in U$ be an arbitrary element
$$Pr[\mathcal{F}' \text{ does not cover } u] = \prod_{S \in \mathcal{F} \text{ s.t } u \in S} (1 - x_S) \le \prod_{S \in \mathcal{F} \text{ s.t } u \in S} e^{-x_S} = e^{-\sum_{S \in \mathcal{F} \text{ s.t } u \in S} x_S} \le e^{-1} = 1/e$$

Idea: Given $\{x_S : S \in \mathcal{F}\}$ make $\mathcal{F}_1', \mathcal{F}_2', \mathcal{F}_3' \dots \mathcal{F}_t'$ for some $t > 1$
Set $\mathcal{F}' = \mathcal{F}_1' \cup \mathcal{F}_2' \cup \mathcal{F}_3' \cup \mathcal{F}_t'$
F' will be bigger, but it might cover G?
- $E[\mathcal{F}'] \le E[\sum_i |\mathcal{F}_i'|] = t \cdot OPT_{LP}$
- $Pr[\mathcal{F}' \text{ does not cover } u] \le e^{-t}$

Set $t = \ln(e \cdot n)$ → $E[|\mathcal{F}'|] \le OPT_{LP}(1 + \ln n)$
$$Pr[\mathcal{F}' \text{ does not cover } u] \le e^{-\ln(en)} = \frac{1}{e \cdot n}$$

We conclude:
$$Pr[C(\mathcal{F}) \neq U] \le Pr[\mathcal{F}' \text{ does not cover } u_1] + Pr[\mathcal{F}' \text{ does not cover } u_2] + \dots Pr[\mathcal{F}' \text{ does not cover } u_n] \le \frac{n}{en} = 1/e$$

UNION BOUND



$Pr[A \vee B] \le Pr[A] + Pr[B]$

This is a $(\ln n + 1)$-APPX FOR SET COVER.

$\boxed{\text{TIGHT !! } (1-\varepsilon)\ln n \Rightarrow P = NP}$ (ask saket!)